Scattering Amplitudes: Theory and Applications

Gravitational Waves

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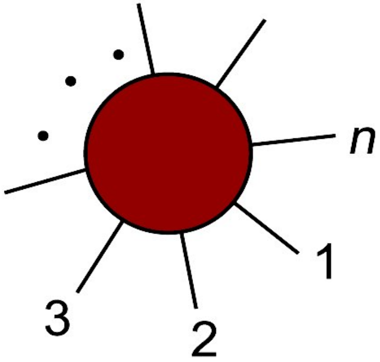
June 19, 2024

Erice School 60th Course:
News from the four interactions

UCLA Mani L. Bhaumik Institute for Theoretical Physics

Outline

Explain some basic modern ideas for scattering and show two application.



1. Spinor Helicity
2. Color Ordering
3. MHV Amplitude
4. On-shell (BCFW) Recursion
5. Generalized Unitarity
6. Color kinematics duality and double copy
7. Brief discussion of UV properties of supergravity
8. Applications to precision gravitational waves.

Reading List: Review Articles

- L. Dixon 1996 TASI lectures hep-ph/9601359 (trees basic ideas)
- Z. Bern, L. Dixon and D. Kosower, hep-ph/9602280 (early review on loops)
- Z. Bern, gr-qc/0206071 (early applications to gravity)
- F. Cachazo and P. Svrcek, hep-th/0504194 (twistors)
- Z. Bern, L. Dixon, D. Kosower, arXiv:0704.2798 (tools for QCD).
- Z. Bern, J.J. Carrasco, M. Chiodaroli, H. Johansson, R. Roiban, arXiv: 1909.01358 (gravity and double copy).
- G. Travaglini et al, arXiv:2203.1301 (The SAGEX review on scattering amplitudes),

Some Terminology

Scattering Amplitude: Quantum field theory description of particle interactions.

QCD: Quantum Chromodynamics. Nonabelian generalization of QED. Theory of gluon and quarks, which are constituents of protons and neutrons.

Supersymmetry: a symmetry relating bosons (forces) to fermions (matter). Central to many topics in phenomenolgy and theory.

$\mathcal{N}$ = 4 super-Yang-Mills: Maximally supersymmetric version of QCD. Links to AdS/CFT.

String Theory: Unified theory of gravity and other force Usually relies on supersymmetry.

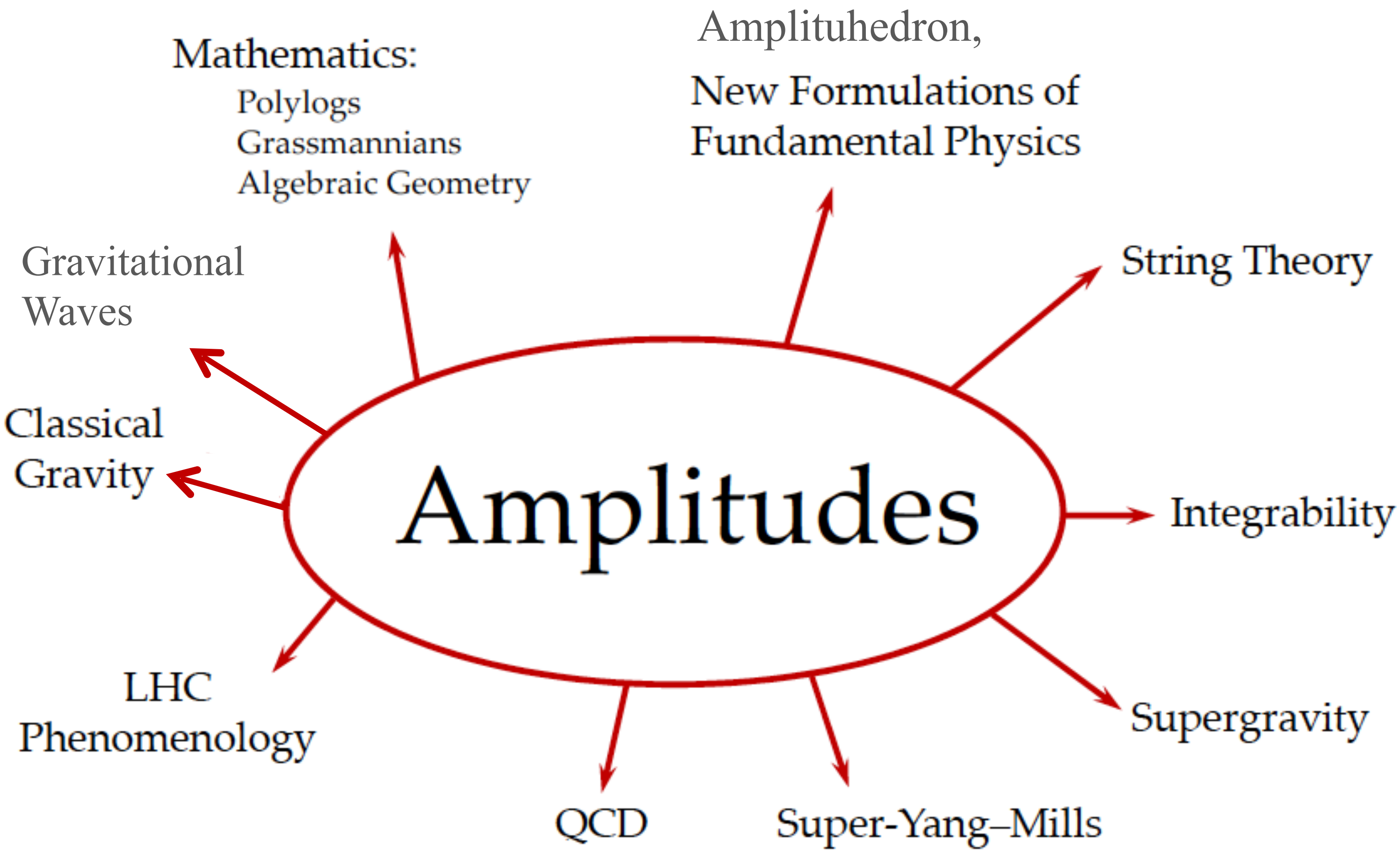
Effective field theory: Very basic tool in quantum field theory, taking advantage of separation of scales.

Unitarity: Quantum evolution must preserve probabilities.

Double copy: The idea that gravitons can be precisely described by two gluons.

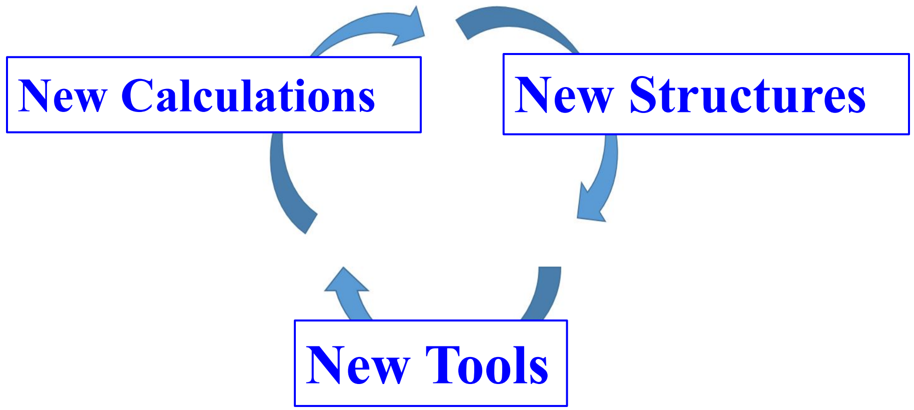
Scattering Amplitudes in QFT

Over time the field has blossomed in many directions.



Vast subject: Here I'm going to pick a few topics.

The search for new structures.



A virtuous circle.

- Key priority for new calculations is to uncover new and useful structures.
- Simultaneously push state of the art for physical quantities of interest.

Three examples of structure that we will discuss:

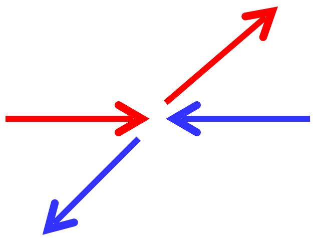
1. Parke-Taylor
2. Curves in Twistor space and MHV rules
3. Double copy

Many other structures, such as amplituhedron

Arkani-Hamed, Trnka

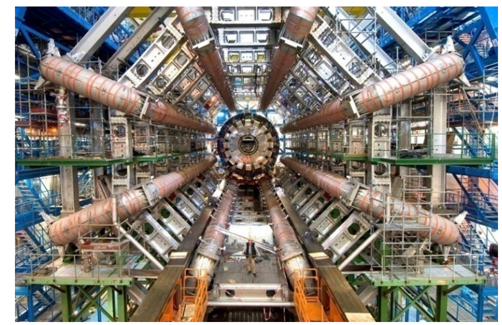
Quantum Field Theory and Scattering Amplitudes

Scattering amplitudes give us quantum mechanical description of events at particle colliders.

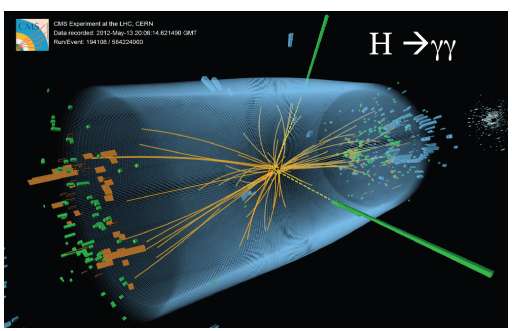


particle scattering

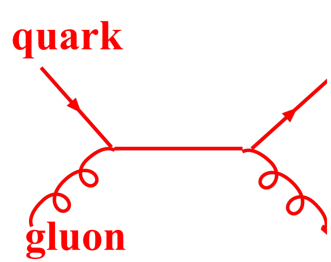
Large Hadron Collider



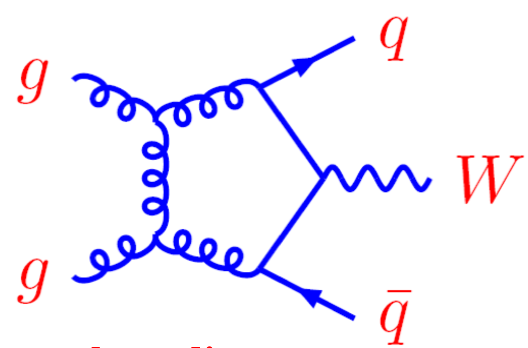
ATLAS Detector



Higgs boson event



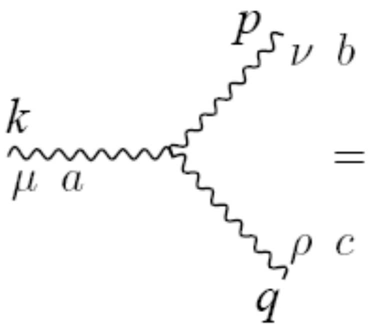
Tree Feynman diagram



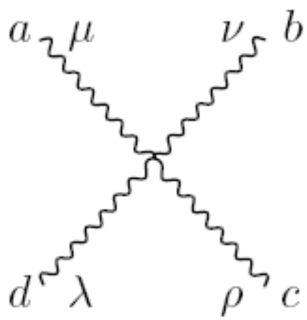
loop diagram
higher order

Scattering amplitudes provide a foundation for understanding of such events.

Gauge Theory Feynman Rules



$$= -g f^{abc}\Big(\eta_{\mu\nu}(k-p)_\rho + \eta_{\nu\rho}(p-q)_\mu + \eta_{\rho\mu}(q-k)_\nu\Big)$$



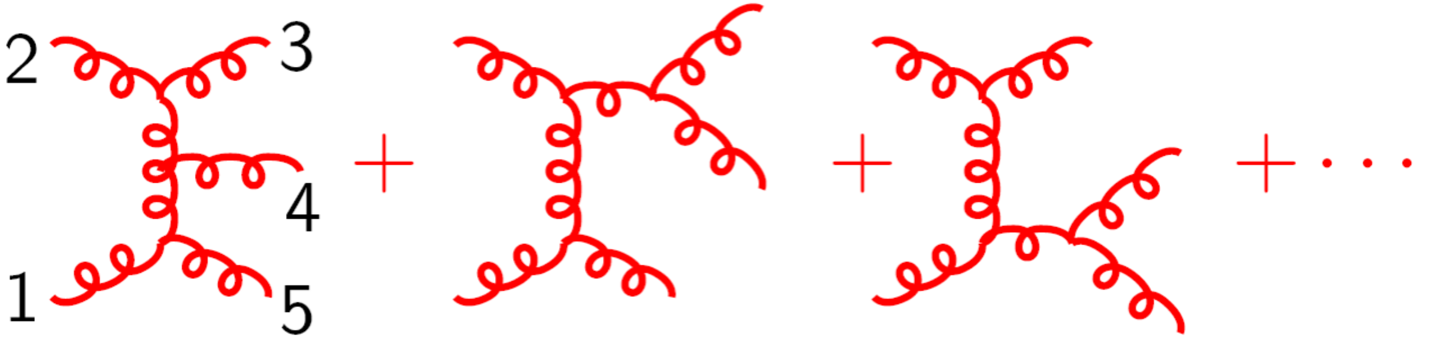
$$= \left\{ \begin{array}{l} -ig^2[f^{abe}f^{ecd}(\eta_{\mu\rho}\eta_{\nu\lambda} - \eta_{\mu\lambda}\eta_{\nu\rho}) \\ \quad + f^{ade}f^{ebc}(\eta_{\mu\nu}\eta_{\rho\lambda} - \eta_{\mu\rho}\eta_{\nu\lambda}) \\ \quad + f^{ace}f^{ebd}(\eta_{\mu\nu}\eta_{\rho\lambda} - \eta_{\mu\lambda}\eta_{\nu\rho})] \end{array} \right.$$

Also fermions and ghosts

Color and kinematics mixed together

Tree-level example: Five gluons

Consider the five-gluon amplitude

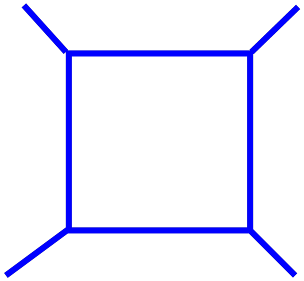


If you evaluate this you find…

Result of evaluation (actually only a small part of it):

$$k_1 \cdot k_4\, \varepsilon_2 \cdot k_1\, \varepsilon_1 \cdot \varepsilon_3\, \varepsilon_4 \cdot \varepsilon_5$$

Example of loop difficulty



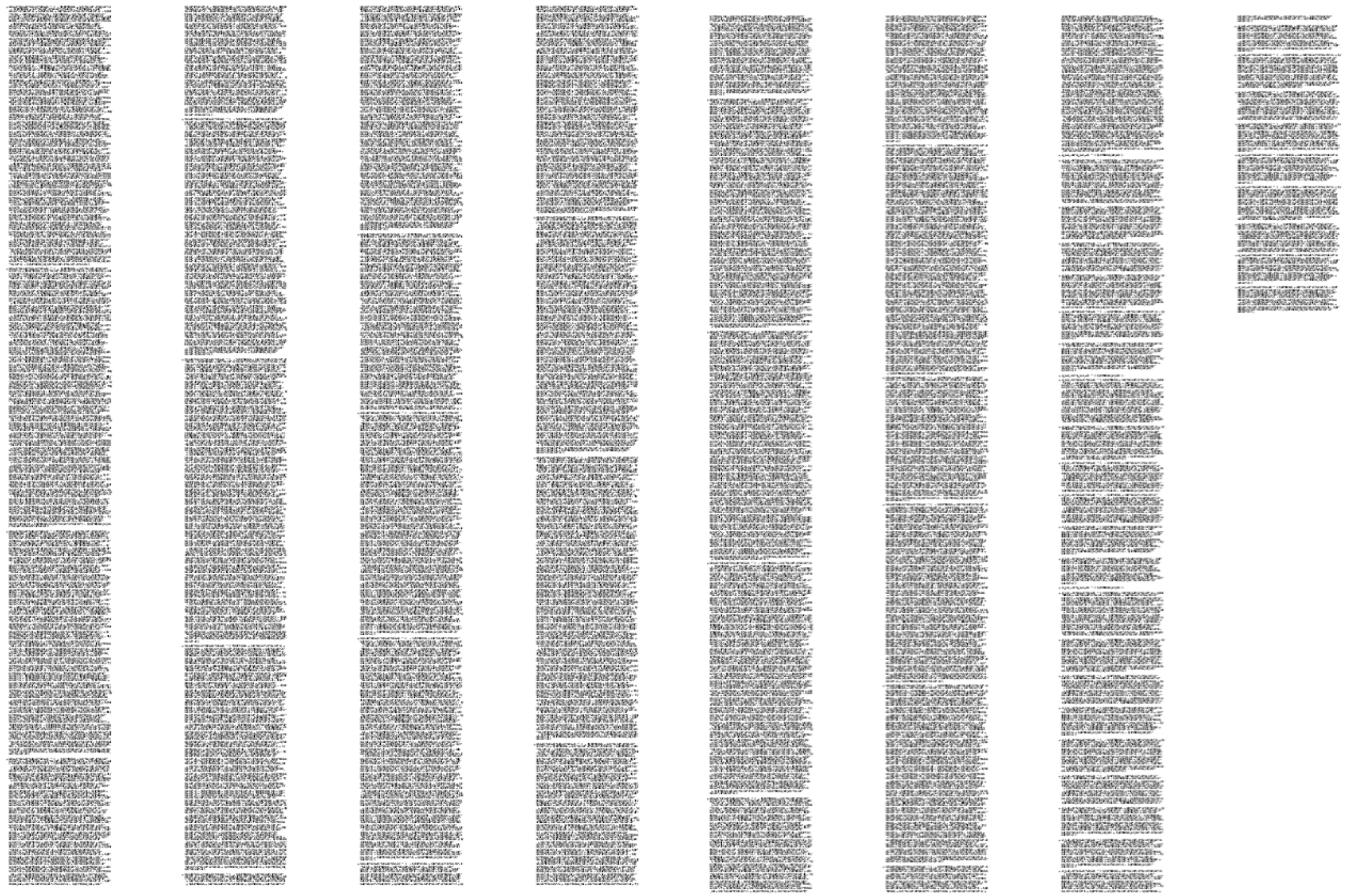
Consider a tensor integral:

$$\int \frac{d^{4-2\epsilon}\ell}{(2\pi)^{4-\epsilon}} \frac{\ell^\mu \, \ell^\nu \, \ell^\rho \, \ell^\lambda}{\ell^2 \, (\ell - k_1)^2 \, (\ell - k_1 - k_2)^2 \, (\ell + k_4)^2}$$

Note: this is trivial on modern computer. Non-trivial for larger numbers of external particles.

Evaluate this integral via Passarino-Veltman reduction. Result is …

Result of performing the integration



Calculations explode for larger numbers of particles or loops. Clearly, there should be a better way!

Spinors expose simplicity

Xu, Zhang and Chang
Berends, Kleis and Causmaeker
Gastmans and Wu
Gunion and Kunszt
& many others

Spinor helicity for massless polarization vectors:

particle momentum

$$\varepsilon^{+}_{\mu}(k;q) = \frac{\langle q^{-}|\gamma_{\mu}|k^{-}\rangle}{\sqrt{2}\,\langle q\,k\rangle}, \qquad \varepsilon^{-}_{\mu}(k,q) = \frac{\langle q^{+}|\gamma_{\mu}|k^{+}\rangle}{\sqrt{2}\,[k\,q]}$$

Reference momentum

Chinese magic

More sophisticated version of circular polarization: $\epsilon_{\mu} = (0, 1, \pm i, 0)$

$$\epsilon^{ab}\lambda_{ja}\lambda_{lb} \longleftrightarrow \langle j\,l\rangle = \langle k_{j\,-}|k_{l+}\rangle = \sqrt{2k_j\cdot k_l}\;e^{i\phi} = \tfrac{1}{2}\bar{u}(k_j)(1+\gamma_5)u(k_l)$$

$$\epsilon_{\dot{a}\dot{b}}\tilde{\lambda}^{\dot{a}}_{j}\tilde{\lambda}^{\dot{b}}_{l} \longleftrightarrow [j\,l] = \langle k_{j\,+}|k_{l\,-}\rangle = -\sqrt{2k_j\cdot k_l}\;e^{-i\phi} = \tfrac{1}{2}\bar{u}(k_j)(1-\gamma_5)u(k_l)$$

All required properties of circular polarization satisfied:

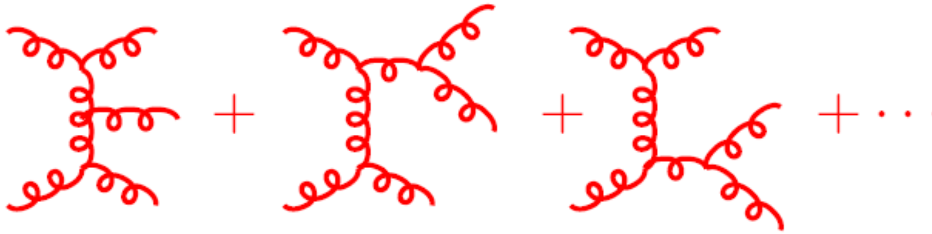
$$\epsilon_i^2 = 0, \qquad k\cdot\epsilon_i = 0, \qquad \epsilon_i^{+}\epsilon_i^{-} = -1$$

Changes in reference momentum *q* equivalent to on-shell gauge transformations. No physical effects.

Graviton polarization tensors are squares of these:

$$\epsilon^{+}_{\mu\nu} = \epsilon^{+}_{\mu}\epsilon^{+}_{\nu} \qquad\qquad 2 = 1 + 1$$

Reconsider Five Gluon Tree



With a little Chinese magic:

$$A_5^{\text{tree}}(1^\pm, 2^+, 3^+, 4^+, 5^+) = 0$$

$$A_5^{\text{tree}}(1^-, 2^-, 3^+, 4^+, 5^+) = i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$

$$A_5^{\text{tree}}(1^-, 2^+, 3^-, 4^+, 5^+) = i\frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$

These are color stripped amplitudes:

$$\mathcal{A}_5 = \sum_{\text{perms}} \text{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}T^{a_5})\, A_5(1,2,3,4,5)$$

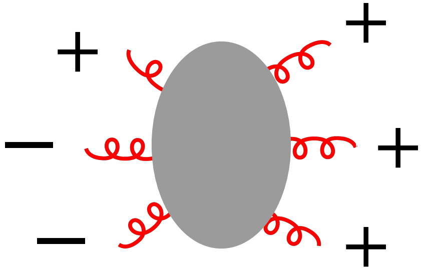
Motivated by the Chan-Paton color organization of open string amplitudes.

Mangano and Parke

Parke and Taylor (1984)
Mangano Parke, Xu

MHV Amplitude

At tree level Parke and Taylor conjectured a very simple form for *n*-gluon scattering.

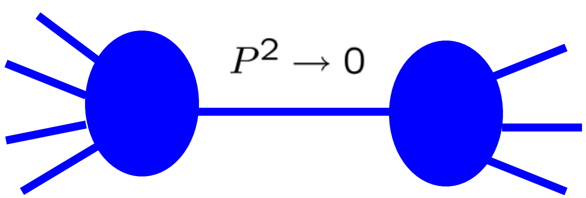


$$A(1^-, 2^-, 3^+, \ldots, n^+) = i \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle \langle 2\,3\rangle \cdots \langle n\,1\rangle}$$

$$\mathcal{A}(1^-, 2^-, 3^+, \ldots, n^+) = \sum_{\text{perms}} \text{Tr}[T^{a_1} T^{a_2} \cdots T^{a_n}] A(1^-, 2^-, 3^+, \ldots, n^+)$$

This was guessed by calculating low points and then finding a formula with correct kinematic poles in all channels.

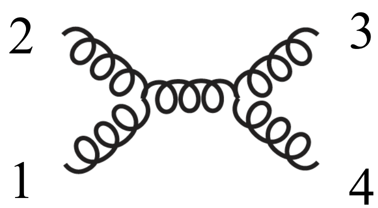
Proven by Berends and Giele



This simplicity has echoes for general helicities and at loop level.

ZB, Dixon, Dunbar, Kosower
Cachazo, Svrcek, Witten; ZB,Dixon, Kosower
Brandhuber, Spence and Travaglini

SU(N_c) Color Decomposition



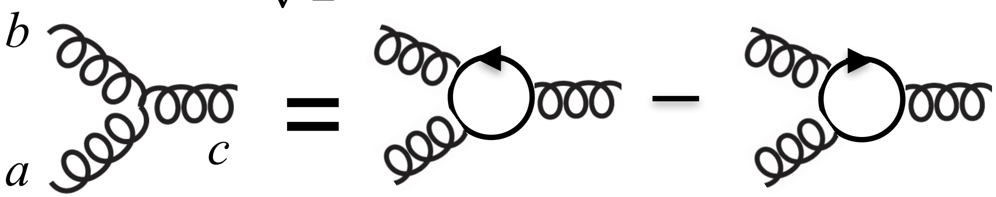
Color factor

$$f^{a_1 a_2 b} f^{a_2 a_3 b}$$

$$\mathrm{Tr}[T^a T^b] = \delta^{ab}$$

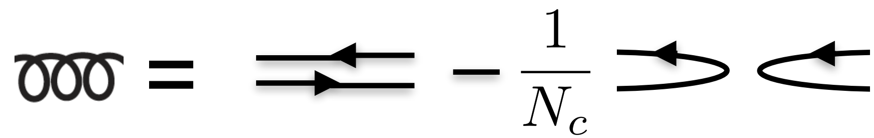
$$[T^a, T^b] = i f^{abc} T^c$$

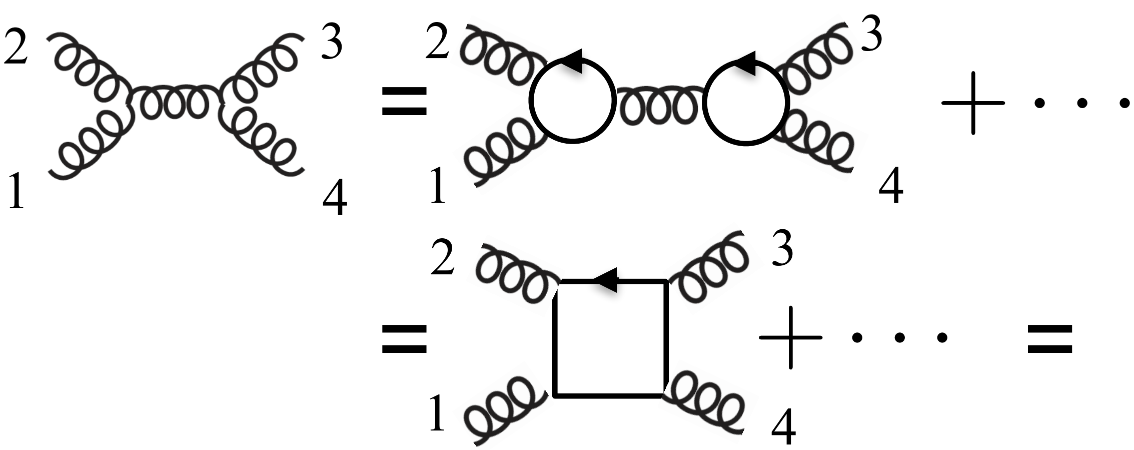
$$f^{abc} = \frac{i}{\sqrt{2}}[\mathrm{Tr}(T^a T^b T^c) - \mathrm{Tr}(T^a T^c T^b)]$$



Fierz identity:

$$(T^a)_{i_1}^{j_1}(T^a)_{i_2}^{j_2} = \delta_{i_1}^{j_2}\delta_{i_2}^{j_1} - \frac{1}{N_c}\delta_{i_1}^{j_1}\delta_{i_2}^{j_2}$$





$= \mathrm{tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) + \cdots$

Use this to reorganize color

SU(N_c) Color Decomposition

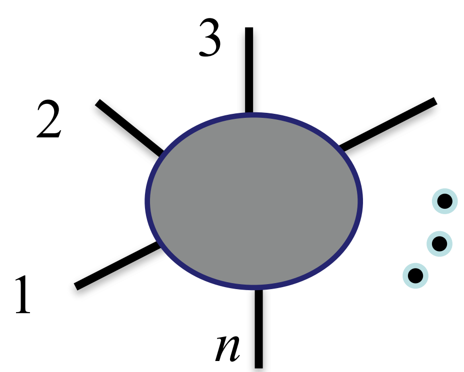
Following above we can prove:

$$\mathcal{A}_n^{\text{tree}} = \sum_{\substack{\text{noncyclic} \\ \text{perms}}} \text{tr}(T^{a_1} T^{a_2} \cdots T^{a_n}) A_n^{\text{tree}}(1, 2, \ldots, n)$$

full amplitude **partial amplitude**

Properties:

- **Partial amplitudes are gauge invariant**
- **Only contributing Feynman diagram follow ordering of legs.**

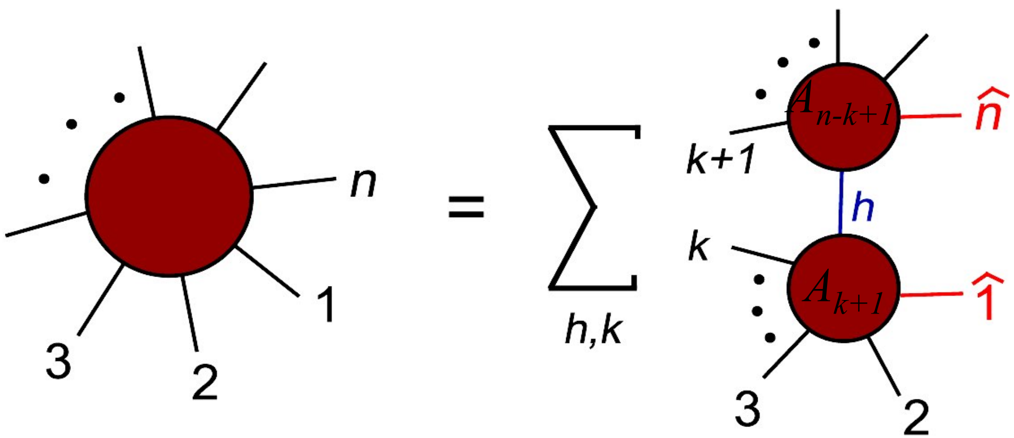


Planar diagrams

On-shell Recursion

A general machinery for constructing tree-level scattering amplitudes is on-shell recursion.

Britto, Cachazo, Feng and Witten



Building blocks are on-shell amplitudes

General replacement for tree-level Feynman diagrams

Contrast with Feynman diagram which are based on off-shell unphysical states with $p^2 \neq m^2$

Britto, Cachazo, Feng and Witten

Proof relies on so little. Power comes from generality.

- **Cauchy's theorem**
- **Basic field theory factorization properties**
- **Applies as well to massive theories.**
- **Applies as well to gravity theories.**

Britto, Cachazo, Feng and Witten
Badger, Glover, Khoze and Svrcek
Bedford, Brandhuber, Spence,Travaglini
Cachazo and Svrcek; Benincasa, Boucher-Veronneau and Cachazo

On-shell Recursion

Consider amplitude under complex deformation of momenta.

$$p_1^\mu \to p_1^\mu - \frac{z}{2}\langle 1^-|\gamma^\mu|n^-\rangle \qquad \tilde\lambda_1 \to \tilde\lambda_1 - z\tilde\lambda_2$$

or

$$p_n^\mu \to p_n^\mu + \frac{z}{2}\langle 1^-|\gamma^\mu|n^-\rangle \qquad \lambda_n \to \lambda_n + z\lambda_1$$

complex momenta $\quad p_i^2 = 0$

Fierz Identity:

$$\langle a^-|\gamma^\mu|b^-\rangle\langle c^-|\gamma_\mu|d^-\rangle = 2\langle ac\rangle[db]$$

$$\langle a^-|\gamma^\mu|b^-\rangle^2 = 0$$

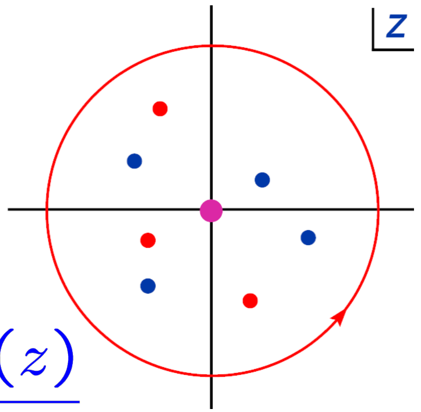
$$\left(p_1 - \frac{z}{2}\langle 1^-|\gamma^\mu|n^-\rangle\right)^2 = p_1^2 + z\langle 1^-|\not{k}_1|n^-\rangle + \frac{z^2}{4}\langle 1^-|\gamma^\mu|n^-\rangle^2$$

$$= 0$$

The shifted momentum are on shell!

On-shell Recursion

$$p_1^\mu(z) = p_1^\mu - \frac{z}{2}\langle 1^-|\gamma^\mu|n^-\rangle \qquad p_n^\mu(z) = p_n^\mu + \frac{z}{2}\langle 1^-|\gamma^\mu|n^-\rangle$$



***A(z)* is amplitude with shifted momenta**

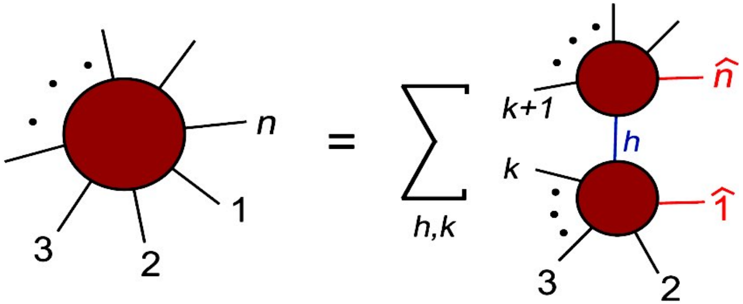
If $A(z) \to 0, \quad z \to \infty$

physical unshifted amplitude

$$\oint_{C_\infty} \frac{A(z)}{z} dz = 0 \quad \Rightarrow A(z=0) = -\sum_\alpha \mathrm{Res}_\alpha \frac{A(z)}{z}$$

$$A(z) = \sum_\alpha \frac{c_\alpha}{z - z_\alpha}$$

Poles in *z* come from kinematic poles



Sum over residues gives the on-shell recursion relation

***h* is helicity**

On-shell Recursion

We can build on-shell amplitudes recursively using simpler on-shell amplitudes.

$$A_4^{\rm tree}(1^-,2^-,3^+,4^+) = i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

$$0 = \oint dz\, \frac{A(z)}{z}$$

Shift

$$\tilde\lambda_1 \to \tilde\lambda_1 - z\tilde\lambda_2 \qquad \lambda_2 \to \lambda_2 + z\lambda_1$$

$$\langle 12\rangle \to \langle 12\rangle + z\langle 11\rangle = \langle 12\rangle$$

$$\langle 23\rangle \to \langle 23\rangle + z\langle 13\rangle$$

Note: $A(z) \to 0$ for $z \to \infty$

Pole at $\quad z_{23} = -\frac{\langle 23\rangle}{\langle 13\rangle}$

$$A_4^{\rm tree}(1,2,3,4) = A_3^{\rm tree}(4^+,\hat 1^-,\hat K_{23}^-)\,\frac{i}{K_{23}^2}\,A_3^{\rm tree}(\hat 2^-,3^+,-\hat K_{23}^+)\Big|_{z=z_{23}}$$

Homework: Prove the Parke-Taylor formula is correct by recursively applying Cauchy's formula.

See Bern, Dixon, Kosower arXiv:0704.2798 sect 2.1 for help

Twistors

Witten demonstrated that twistor space reveals a hidden structure in scattering amplitudes.

Link is for $\mathcal{N} = 4$ super-Yang-Mills theory, but at tree level hardly any difference from QCD.

Penrose twistor transform:

Early work from Nair

$$\widetilde{A}(\lambda_i, \mu_i) = \int \prod_i \frac{d^2\widetilde{\lambda}_i}{(2\pi)^2} \exp\left(\sum_j i\mu_j^{\dot{a}} \widetilde{\lambda}_{j\dot{a}} \right) A(\lambda_i, \widetilde{\lambda}_i)$$

Witten's remarkable twistor-space link:

Witten; Roiban, Spradlin and Volovich

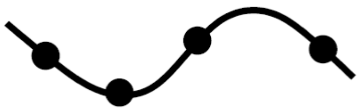
QCD scattering amplitudes $\longleftrightarrow$ Topological String Theory

Here we will discuss only the field theory consequences

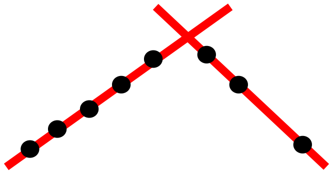
Amazing Structures

Witten conjectured that in twistor–space gauge theory amplitudes have delta-function support on curves of degree:

$$d = q - 1 + L, \quad q = \#\ \text{negative helicities}, \quad L = \#\ \text{loops},$$



Connected picture



Disconnected picture

Witten
Roiban, Spradlin and Volovich
Cachazo, Svrcek and Witten
Gukov, Motl and Neitzke
Bena Bern and Kosower

Structures imply an amazing simplicity in the scattering amplitudes.

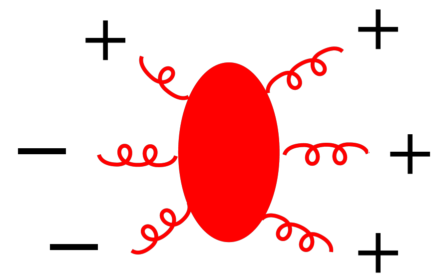
Remarkably gravity is similar, except derivative of delta function support instead of delta-function support.

MHV Rules

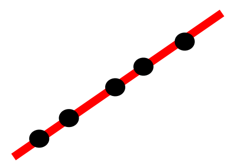
Cachazo, Svrcek and Witten

Consider MHV amplitudes.

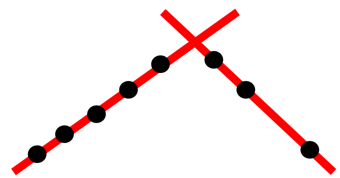
$$A(1^-,2^-,3^+,\ldots,n^+) = i\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle n\,1\rangle}$$

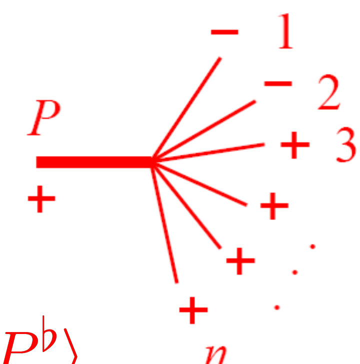


Supported on a straight line in twistor space

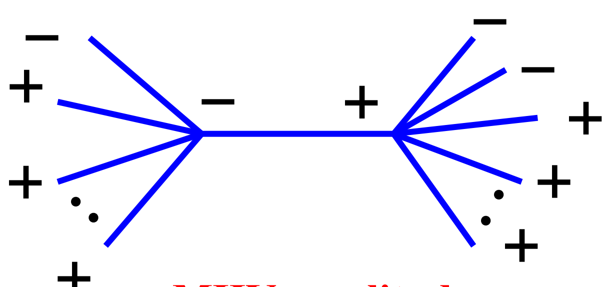


Non-MHV amplitudes supported on intersecting lines





In momentum space suggests MHV amplitudes are vertices for building new amplitudes.



MHV amplitudes at vertices

$\langle a P\rangle \to \langle a|\not{P}|q\rangle$ or $\langle a P\rangle \to \langle a P^\flat\rangle$

$$q^2 = 0$$

Arbitrary null momentum

$$P^\flat \equiv P - \frac{P^2}{P\cdot q}\,q$$

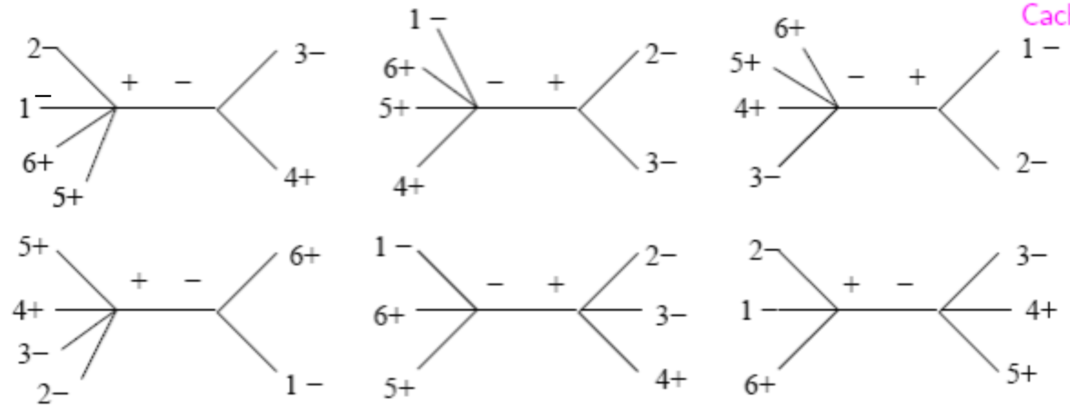
Massless

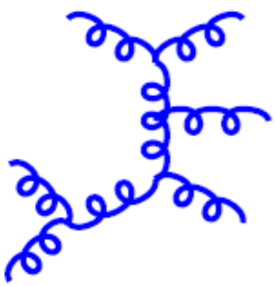
Six-gluon example

Cachazo, Svrcek and Witten

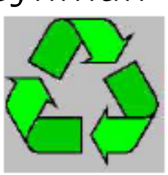
QCD gluon scattering amplitude

$- - - + + +$





220 Feynman diagrams



$$
\begin{aligned}
A_6(1^-,2^-,3^-,4^+,5^+,6^+) = {}& \frac{\langle 1\,2\rangle^3}{\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 2|\,5+6+1\,|q\rangle\,\langle 5|\,6+1+2\,|q\rangle} \times \frac{1}{s_{34}} \times \frac{\langle 3|\,4\,|q\rangle^3}{\langle 3\,4\rangle\,\langle 4|\,3\,|q\rangle} \\
&+ \frac{\langle 1|\,4+5+6\,|q\rangle^3}{\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 4|\,5+6+1\,|q\rangle} \times \frac{1}{s_{23}} \times \frac{\langle 2\,3\rangle^3}{\langle 3|\,2\,|q\rangle\,\langle 2|\,3\,|q\rangle} \\
&+ \frac{\langle 3|\,4+5+6\,|q\rangle^3}{\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6|\,3+4+5\,|q\rangle} \times \frac{1}{s_{12}} \times \frac{\langle 1\,2\rangle^3}{\langle 2|\,1\,|q\rangle\,\langle 1|\,2\,|q\rangle} \\
&+ \frac{\langle 2\,3\rangle^3}{\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5|\,2+3+4\,|q\rangle\,\langle 2|\,3+4+5\,|q\rangle} \times \frac{1}{s_{61}} \times \frac{\langle 1|\,6\,|q\rangle^3}{\langle 6\,1\rangle\,\langle 6|\,1\,|q\rangle} \\
&+ \frac{\langle 1|\,5+6\,|q\rangle^3}{\langle 5\,6\rangle\,\langle 6\,1\rangle\,\langle 5|\,6+1\,|q\rangle} \times \frac{1}{s_{561}} \times \frac{\langle 2\,3\rangle^3}{\langle 3\,4\rangle\,\langle 4|\,2+3\,|q\rangle\,\langle 2|\,3+4\,|q\rangle} \\
&+ \frac{\langle 1\,2\rangle^3}{\langle 6\,1\rangle\,\langle 2|\,6+1\,|q\rangle\,\langle 6|\,1+2\,|q\rangle} \times \frac{1}{s_{612}} \times \frac{\langle 3|\,4+5\,|q\rangle^3}{\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5|\,3+4\,|q\rangle}
\end{aligned}
$$

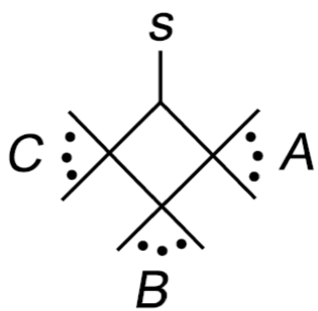
A “perfect” calculation

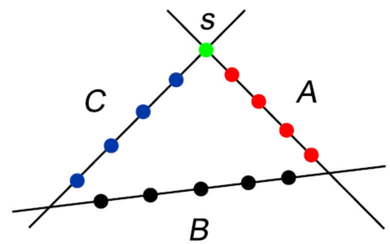
Twistor Structure at One Loop

At one-loop the coefficients of all integral functions have beautiful twistor space interpretations

Box integral | **Twistor space support**

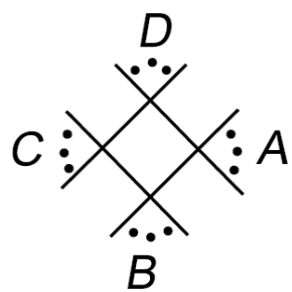
Three negative helicities

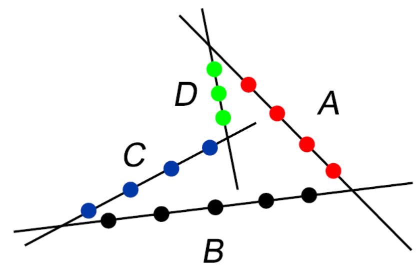




Bern, Dixon and Kosower
Britto, Cachazo and Feng

Four negative helicities





The existence of such twistor structures connected with loop-level simplicity.
Higher-loop structures

Remarkable Twistor String Formula

The following formula encapsulates the entire tree-level S-matrix of $\mathcal{N} = 4$ super-Yang-Mills (or pure gluons in QCD)

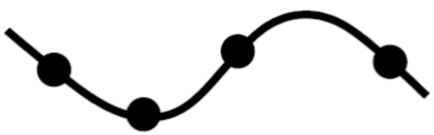
Witten
Roiban, Spradlin and Volovich

Integral over the Moduli and curves

$$A_n = i(2\pi)^4 g_{\rm YM}^{n-2} \sum_{d=1}^{n-3} \int d\mathcal{M}_{n,d} \prod_{i=1}^{n} \delta^2\left(\lambda_i^\alpha - \xi_i P_i^\alpha\right) \prod_{k=0}^{d} \delta^2\left(\sum_{i=1}^{n} \xi_i \sigma_i^k \widetilde{\lambda}_i^{\dot\alpha}\right) \delta^4\left(\sum_{i=1}^{n} \xi_i \sigma_i^k \eta_{iA}\right)$$

$$P_i^\alpha = \sum_{k=0}^{d} a_k^\alpha \sigma_i^k$$

Degree d polynomial in the σ_i

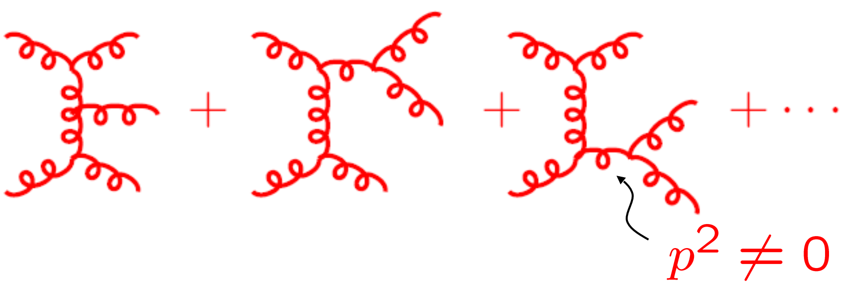


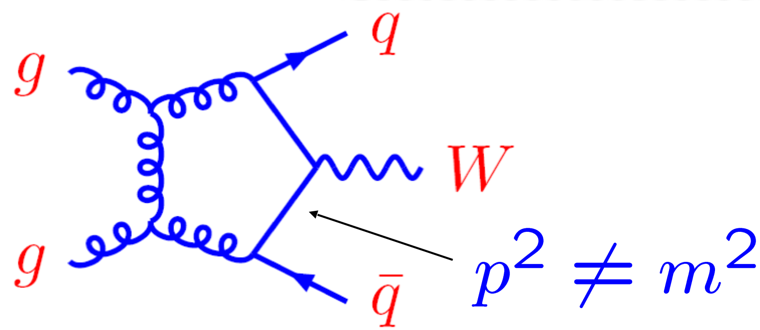
Very strange formula from Feynman diagram viewpoint. Perfect example of a hidden structure.

But it's true: impressive checks by Roiban, Spradlin and Volovich

Why are Feynman diagrams clumsy for high loop or multiplicity processes?

- Vertices and propagators involve gauge-dependent off-shell states. Origin of the complexity.





- To get at root cause of the trouble we must rewrite perturbative quantum field theory.

- **All steps should be in terms of gauge invariant on-shell states.** $p^2 = m^2$ **On shell formalism.**
- **Radical rewrite of gauge theory needed.**

Surprising Simplicity of Pure Gluon Amplitudes

ZB, Dixon, Kosower 1993

$$A_{5;1}^{[0]} = c_\Gamma \left(V^s A_5^{\rm tree} + iF^s\right) \,,$$
$$A_{5;1}^{[1/2]} = -c_\Gamma \left((V^f + V^s)A_5^{\rm tree} + i(F^f + F^s)\right) \,,$$
$$A_{5;1}^{[1]} = c_\Gamma \left((V^g + 4V^f + V^s)A_5^{\rm tree} + i(4F^f + F^s)\right)$$

Supersymmetric decomposition of QCD

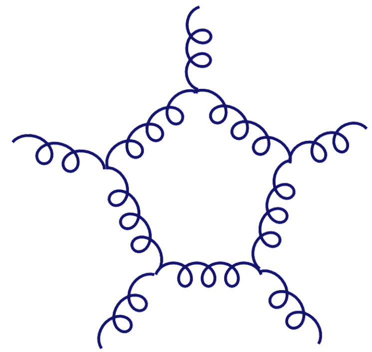
$$V^g = -\frac{1}{\epsilon^2}\sum_{j=1}^{5}\left(\frac{\mu^2}{-s_{j,j+1}}\right)^{\epsilon} + \sum_{j=1}^{5}\ln\left(\frac{-s_{j,j+1}}{-s_{j+1,j+2}}\right)\ln\left(\frac{-s_{j+2,j-2}}{-s_{j-2,j-1}}\right) + \frac{5}{6}\pi^2$$

***N* = 4 sYM**

$$V^f = -\frac{5}{2\epsilon} - \frac{1}{2}\left[\ln\left(\frac{\mu^2}{-s_{23}}\right) + \ln\left(\frac{\mu^2}{-s_{51}}\right)\right] - 2, \qquad V^s = -\frac{1}{3}V^f + \frac{2}{9}$$

$$F^f = -\frac{1}{2}\frac{\langle 1\,2\rangle^2 \left(\langle 2\,3\rangle [3\,4] \langle 4\,1\rangle + \langle 2\,4\rangle [4\,5] \langle 5\,1\rangle\right)}{\langle 2\,3\rangle \langle 3\,4\rangle \langle 4\,5\rangle \langle 5\,1\rangle}\frac{\mathrm{L}_0\left(\frac{-s_{23}}{-s_{51}}\right)}{s_{51}}$$

$$F^s = -\frac{1}{3}\frac{[3\,4] \langle 4\,1\rangle \langle 2\,4\rangle [4\,5] \left(\langle 2\,3\rangle [3\,4] \langle 4\,1\rangle + \langle 2\,4\rangle [4\,5] \langle 5\,1\rangle\right)}{\langle 3\,4\rangle \langle 4\,5\rangle}\frac{\mathrm{L}_2\left(\frac{-s_{23}}{-s_{51}}\right)}{s_{51}^3} - \frac{1}{3}F^f$$
$$- \frac{1}{3}\frac{\langle 3\,5\rangle [3\,5]^3}{[1\,2] [2\,3] \langle 3\,4\rangle \langle 4\,5\rangle [5\,1]} + \frac{1}{3}\frac{\langle 1\,2\rangle [3\,5]^2}{[2\,3] \langle 3\,4\rangle \langle 4\,5\rangle [5\,1]} + \frac{1}{6}\frac{\langle 1\,2\rangle [3\,4] \langle 4\,1\rangle \langle 2\,4\rangle [4\,5]}{s_{23} \langle 3\,4\rangle \langle 4\,5\rangle s_{51}}$$



Amazingly it fit into PRL

Well defined pieces of one-loop QCD are supersymmetric and simple. There must be a better way.

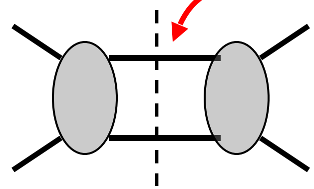
Modern Unitarity Method

$$E^2 = \vec{p}^{\,2} + m^2$$

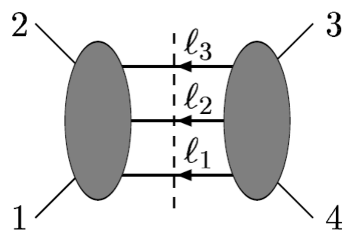
on-shell

ZB, Dixon, Dunbar and Kosower

Two-particle cut:



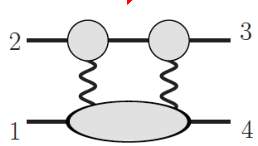
Three-particle cut:

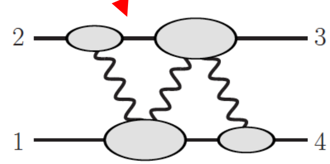


- **Systematic assembly of complete loop amplitudes from tree amplitudes.**
- **Works for any number of particles or loops.**

on-shell

Generalized unitarity as a practical tool.





ZB, Dixon and Kosower;
ZB, Morgan;
Britto, Cachazo, Feng;
Ossala,Pittau,Papadopoulos;
Ellis, Kunszt, Melnikov;
Forde; Badger
and many others

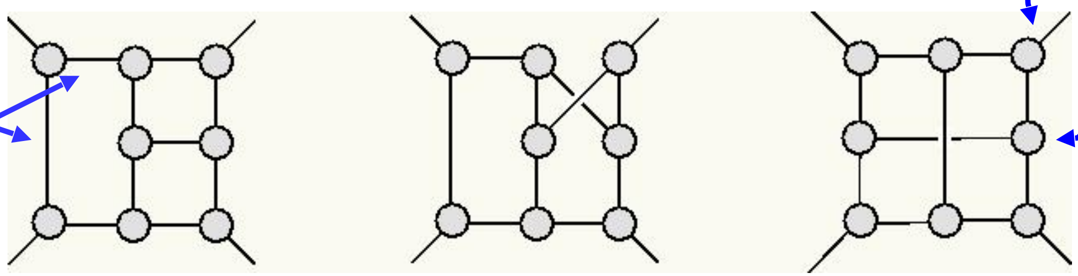
Idea used in the "NLO revolution" in QCD collider physics and high loop supergravity calculations
Are applying it to gravitational-wave problem.

Method of Maximal Cuts

ZB, Carrasco, Johansson, Kosower

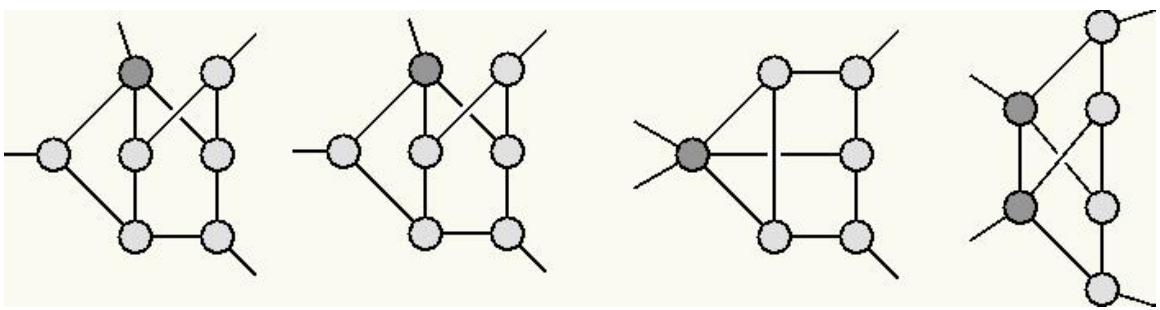
A refinement of unitarity method is "Method of Maximal Cuts".

To construct the amplitude we use cuts with maximum number of on-shell propagators:



Maximum number of propagator placed on-shell.

Then systematically release cut conditions to obtain contact terms:



Fewer propagators placed on-shell.

Effectively this picks out terms in the integrand.

Slick organization compatible with double copy: ZB, Herrmann, Roiban, Ruf, Zeng (arXiv:2406.XXXX)

Summary

- **Scattering amplitudes simpler than anyone imagined.**
- **On-shell methods exploit the simplicity.**
- **Unitarity method gives a means for exploiting tree-level simplicity to construct higher-loop amplitudes.**

In next lecture we will discuss remarkable double-copy property of gravity and start discussing applications.

Reading List: Review Articles

- L. Dixon 1996 TASI lectures hep-ph/9601359 (trees basic ideas)

$$\tilde{\lambda}_1 \to \tilde{\lambda}_1 - z\tilde{\lambda}_2$$

- Z. Bern, L. Dixon and D. Kosower, hep-ph/9602280 (early review on loops)
- Z. Bern, gr-qc/0206071 (early applications to gravity)
- F. Cachazo and P. Svrcek, hep-th/0504194 (twistors)
- Z. Bern, L. Dixon, D. Kosower, arXiv:0704.2798 (tools for QCD).
- Z. Bern, J.J. Carrasco, M. Chiodaroli, H. Johansson, R. Roiban, arXiv:1909.01358 (gravity and double copy).
- G. Travaglini et al, arXiv: 2203.1301 (The SAGEX review on scattering amplitudes),