Harmonic superspace: an essential tool for extended supersymmetries

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Question: How to construct appropriate superspace for $\mathcal{N}$ extended supersymmetry theories in terms of extrended superfields with all the symmetries being manifest?

Why?

1. Study of higher dimensional supersymmetric field theories is associated with superstring theory.
2. Specific feature of the superstring theory is existence of so called D branes which are the $D+1$ dimensional surfaces in the ten-dimensional space-time. In the low-energy limit the D brane is associated with $D+1$-dimensional supersymmetric gauge theory. Therefore, study of low-energy limit of superstring theory can be related to (extended) supersymmetric field theory in various dimensions.
3. Study of quantum field models with large number of symmetries: gauge symmetry, global symmetries, supersymmetries.
4. Quantization procedure with preservation of all explicit symmetries. Perturbation theory with preservation of all explicit symmetries. Hidden (on-shell) symmetries. Preservation of hidden symmetries.
5. Construction of the new extended supersymmetric invariants as the the quantum contributions to effective action.

$\mathcal{N}$-extended supersymmetry

The $\mathcal{N}$ Poincaré supersymmetry, along with the standard Poincaré group generators P_m, J_{mn} ($m, n = 0, 1, 2, 3$; P_m – are the 4-translation generators and J_{mn} – the Lorentz group ones), involves the fermionic generators $Q^I_\alpha, \bar{Q}^I_{\dot\alpha}$, where $\alpha, \dot\alpha = 1, 2$; and $I = \ldots \mathcal{N}$.

$$\begin{aligned} [P_m, Q^I_\alpha] &= 0, \quad [P_m, \bar{Q}^I_{\dot\alpha}] = 0, \\ [J_{mn}, Q^I_\alpha] &= i(\sigma_{mn})^\beta_\alpha Q^I_\beta, \quad [J_{mn}, \bar{Q}^I_{\dot\alpha}] = i(\tilde\sigma_{mn})^{\dot\beta}_{\dot\alpha} \bar{Q}^I_{\dot\beta} \\ \{Q^I_\alpha, Q^J_\beta\} &= \varepsilon_{\alpha\beta} Z^{IJ}, \quad \{\bar{Q}^I_{\dot\alpha}, \bar{Q}^J_{\dot\beta}\} = \varepsilon_{\dot\alpha\dot\beta} \bar{Z}^{IJ}, \\ \{Q^I_\alpha, \bar{Q}^J_{\dot\alpha}\} &= 2\delta^{IJ} (\sigma^m)_{\alpha\dot\alpha} P_m. \end{aligned} \tag{1}$$

Here $\sigma^{mn} = -\frac{1}{4}(\sigma^m\tilde\sigma^n - \sigma^n\tilde\sigma^m)$, $\tilde\sigma^{mn} = -\frac{1}{4}(\tilde\sigma^m\sigma^n - \tilde\sigma^n\sigma^m)$ $\sigma, \tilde\sigma$ – Pauli matrices.
The natural way to realize $\mathcal{N}$-extended Poincaré supersymmetry is to use the standard superspace

$$\mathbb{R}^{4|4N} = (x^m, \theta^\alpha_I, \bar\theta^{\dot\alpha I}), \qquad I = 1, 2, \ldots, N \tag{2}$$

involving the spinor *anticommuting* coordinates $\theta^\alpha_I, \bar\theta^{\dot\alpha I}$, in addition to the commuting x^m.
$\mathcal{N}$-extended supersymmetry can also be realized in the *chiral* superspace $\mathbb{C}^{4|2N}$

$$x^m_L = x^m + i\theta^I \sigma^a \bar\theta_I , \quad \theta^\alpha_{LI} = \theta^\alpha_I . \tag{3}$$

Example $\mathcal{N} = 1$

$\mathbb{C}^{4|2} = (\zeta_L^M) \equiv (x_L^m, \theta_L^\alpha)$. The corresponding superfield

$$\Phi(\zeta_L) = \varphi(x_L) + \theta^\alpha \psi_\alpha(x_L) + \theta\theta F(x_L), \tag{4}$$

where $\varphi(x_L), \psi_\alpha(x_L)$ are physical fields and $F(x_L)$ is an auxiliary field.
The model in $\mathbb{C}^{4|2}$

$$S = \frac{1}{\xi^2} \int d^4x d^2\theta d^2\bar{\theta} K(\Phi, \bar{\Phi}) + \frac{1}{\xi^2} \int d^4x d^2\theta P(\Phi) + \text{c.c.}. \tag{5}$$

Why standard superspace is not adequate for $\mathcal{N}=2$ supersymmetry?

1. 'Not adequate' superspace means that we can not write the expansion $q^I(x,\theta,\bar\theta)=\dots$ defined in $\mathbb{R}^{4|8}$. It could be shown on the example of the Fayet–Sohnius matter hypermultiplet.
2. We have components on shell: four scalar fields forming an $SU(2)$ doublet f^I and two isosinglet spinor fields $\psi_\alpha, \bar k_{\dot\alpha}$. Due to the large number of spinor variables this superfield contains a lot of redundant field components in addition to the physical ones.
3. We can exclude the auxiliary fields with the help of the constraints

$$D_\alpha^{(I}q^{J)}=\bar D_{\dot\alpha}^{(I}q^{J)}=0, \quad \{D_\alpha,\bar D_{\dot\alpha}\}=-2i(\sigma^m)_{\alpha\dot\alpha}\partial_m, \tag{6}$$

then

$$q^I(x,\theta,\bar\theta)=f^I(x)+\theta^{I\alpha}\psi_\alpha(x)+\bar\theta^I_{\dot\alpha}\bar k^{\dot\alpha}(x)+\dots. \tag{7}$$

4. At the same time the above constraints put all the physical fields on the free mass shell

$$\Box f^I(x)=(\sigma^m)^{\alpha\dot\alpha}\partial_m\psi_\alpha(x)=(\sigma^m)_{\alpha\dot\alpha}\partial_m\bar k^{\dot\alpha}(x)=0. \tag{8}$$

Problem solved

1. The standard real superspaces $\mathbb{R}^{4|4N}$ could be represented as

$$\mathbb{R}^{4|4N} = \frac{\mathcal{SP}_{\mathcal{N}}}{\mathcal{L}} = (x_m, \theta^{\alpha}_i, \bar{\theta}^{\dot{\alpha} i}) \tag{9}$$

where $\mathcal{SP}_{\mathcal{N}}$ is the $\mathcal{N}$-extended super-Poincaré generators involving the spinor supersymmetry generators $Q^I_{\alpha}, \bar{Q}^I_{\dot{\alpha}}$. All the nominator means the group for corresponding algebras. In the denominators stand the Lorentz generators in the same manner.

2. Complex (chiral) superspace $\mathbb{C}^{4|2N}$ could be represented the same.
3. The $\mathcal{N} = 2$ superalgebra

$$\{Q^i_{\alpha}, \bar{Q}^j_{\dot{\alpha}}\} = 2\delta^{ij}(\sigma^m)_{\alpha\dot{\alpha}} P_m, i, j = 1, 2, \tag{10}$$

possesses an $SU(2)$ group of automorphisms (R-symmetry). In the standard case this $SU(2)$ can be viewed as present both in the numerator and the denominator.

What happens if we keep only the $U(1)$ subgroup of $SU(2)$ in the denominator:

$$\mathbb{H}^{4+2|8} = \frac{\mathcal{SP}_{\mathcal{N}}}{\mathcal{L}} \times \frac{SU(2)}{U(1)} = (x_m, \theta^{\alpha}_i, \bar{\theta}^{\dot{\alpha} i}, u^{\pm i}),$$
$$(u^+_i, u^-_i) \in S^2, u^{+i} u^-_i = 1. \tag{11}$$

Harmonic superspace happens.

(**A. Galperin, E. Ivanov, V. Ogievetsky, E. Sokatchev** (**1985**)).

1. The main advantage of harmonic superspace is the existence of an invariant subspace in it, the $\mathcal{N}=2$ analytic harmonic superspace with half of the original odd coordinates

$$\mathbb{H}A^{(4+2|4)} = \left(x_A^m, \theta_A^{+\alpha}, \bar{\theta}_A^{+\dot{\alpha}}, u^{\pm i}\right) \equiv \left(\zeta, u_i^{\pm}\right) , \tag{12}$$

$$x_A^m = x^m - 2i\theta^{(i}\sigma^m\bar{\theta}^{k)}u_i^+u_k^- , \quad \theta^{+\alpha} = \theta^{\alpha\, i}u_i^+ , \; \bar{\theta}^{+b\dot{\alpha}} = \bar{\theta}^{\dot{\alpha}\, i}u_i^+ . \tag{13}$$

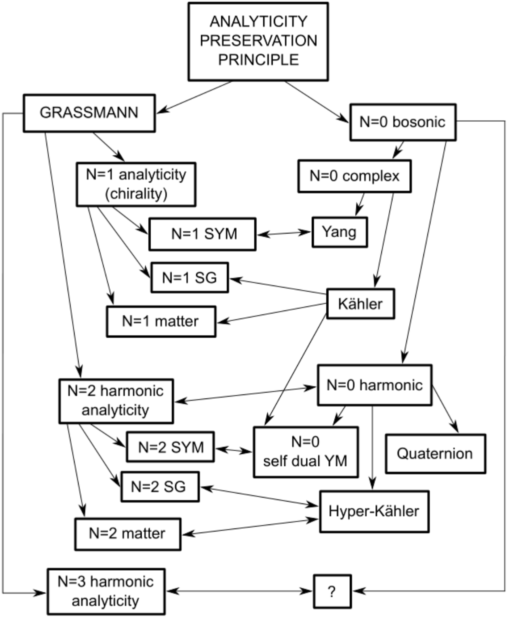
2. It is closed under $\mathcal{N}=2$ supersymmetry transformations and is real with respect to the special involution which is the product of the ordinary complex conjugation and the antipodal map (Weyl reflection) of S^2.
3. All $\mathcal{N}=2$ supersymmetric theories admit off-shell formulations in terms of unconstrained superfields defined on (12), the *Grassmann analytic* $\mathcal{N}=2$ superfields.

4. Fields on S^2 must possess a *definite* $U(1)$ charge q and, as a consequence, all the terms in their harmonic expansion must contain only products of u^+ , u^- For instance, for $q=+1$

$$f^+(u) = f^i u_i^+ + f^{(ijk)}u_{(i}^+u_j^+u_{k)}^- + \dots , \tag{14}$$

5. The superfield expansion is now have only finite number of auxiliary fields along with physical ones. Dut all the components now are harmonic dependant

$$q^+(\zeta,u) = F^+(x,u) + \theta^{+\alpha}\psi_\alpha(x,u) + \bar{\theta}^+\bar{k}^{\dot{\alpha}}(x,u) + \dots . \tag{15}$$


ANALYTICITY PRESERVATION PRINCIPLE
GRASSMANN
N=0 bosonic
N=1 analyticity (chirality)
N=0 complex
N=1 SYM
Yang
N=1 SG
Kähler
N=1 matter
N=0 harmonic
N=2 harmonic analyticity
N=2 SYM
N=0 self dual YM
Quaternion
N=2 SG
Hyper-Kähler
N=2 matter
N=3 harmonic analyticity
?

What is done

$D = 3$

$3d, N = 6$ superconformal theories: correlation functions, BPS operators (**P. Liendo, C. Meneghelli, V. Mitev (2016)**) ; $N = 3, N = 4$ HSS (**B. M. Zupnik (1998)**),

$D = 4$

$N = 2$ matter hypermultiplets, $N = 2, 3$ SYM, $N = 2$ SUGRA, $N = 2$ Higher spins
Basic literature:

1. (**A. Galperin, E. Ivanov, V. Ogievetsky, E. Sokatchev (1985)**)
2. (**A. Galperin, E. Ivanov, V. Ogievetsky, E. Sokatchev - Harmonic Superspace (2001)**)

$D = 5$

$N = 1$ AdS SUSY (**S. M. Kuzenko, G. T. Mazzucchelli (2007)**), O(5) $N = 2$ SYM, (**I.L. Buchbinder, E.A. Ivanov, I.B. Samsonov (2019)**)

$D = 6$

$N = (1, 0)$ SYM, HD SUSY
Basic literature:

1. (**P.S. Howe, K.S. Stelle, P.C. West (1985)**),
2. (**B.M. Zupnik (1986; 1999)**),
3. (**G. Bossard, E. Ivanov, A. Smilga, JHEP (2015)**).

Problem solved?

1. Nonlinear version of $\mathcal{N}=2$ higher spins,
2. $\mathcal{N}=3$ supergravity in $\mathcal{N}=3$ HSS,
3. $\mathcal{N}=8$ SUGRA finiteness?
4. $\mathcal{N}=4$ superspace formulation of four-dimensional SUSY theories,
5. $\mathcal{N}=(1,1)$ superspace formulation of six-dimensional SUSY theories, e.t.c.

Is there any connection between R-symmetry and corresponding algebraic manifold $(SU(?))$?

Thank you for your attention!