Scattering Amplitudes: Theory and Applications
Double Copy

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June 20, 2024

Erice School 60th Course:
News from the four interactions

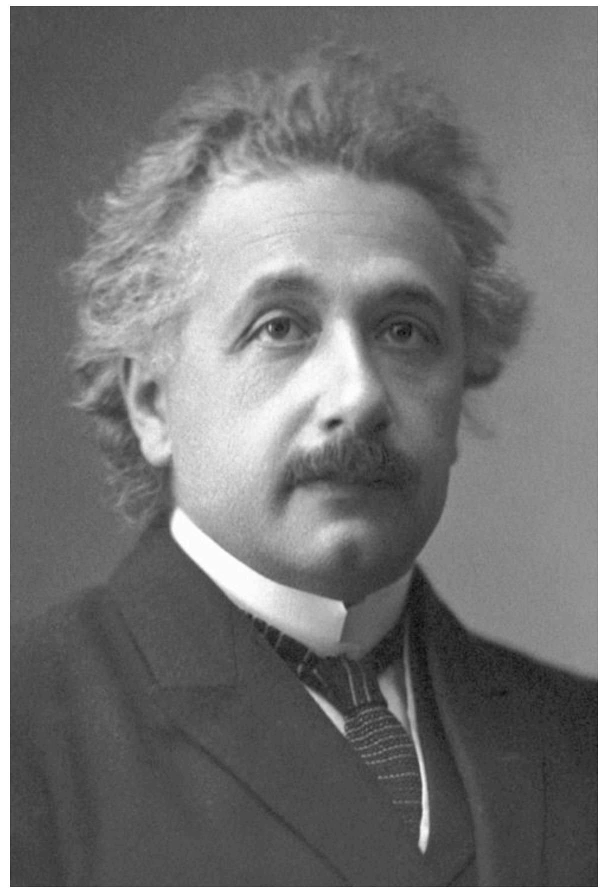
UCLA Mani L. Bhaumik Institute for Theoretical Physics

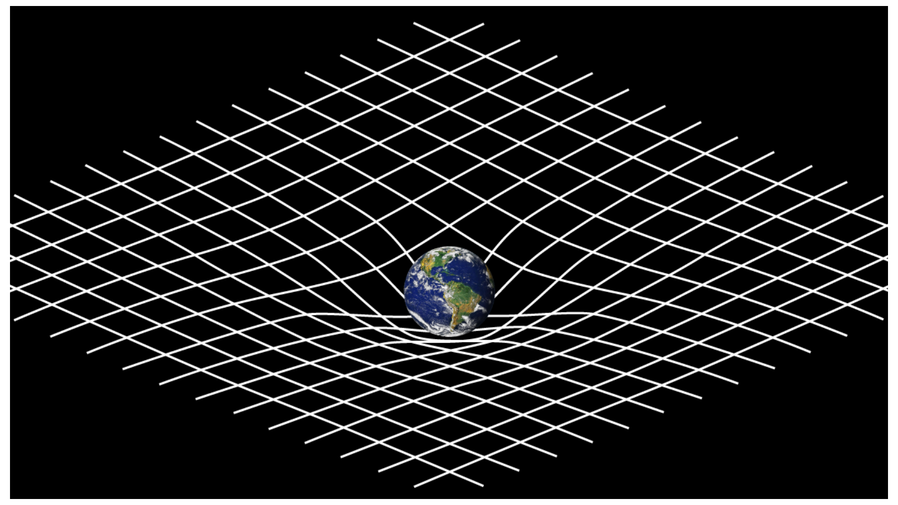
Approach to General Relativity

Our appoach does *not* start from usual Einstein Field equations.

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$$

geometry



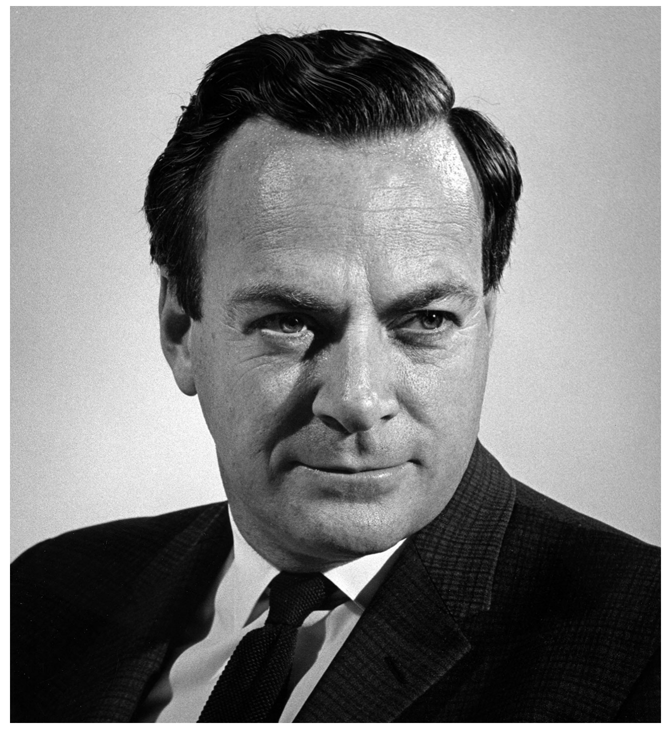


Amplitude Approach to General Relativity

Our appoach does *not* start from usual Einstein Field equations.

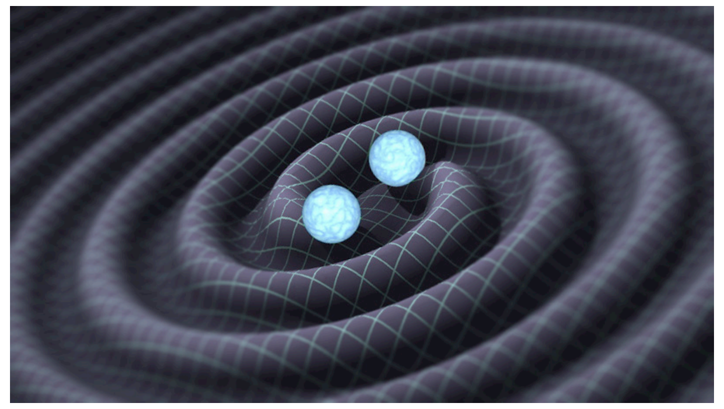
$$\cancel{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}}$$

~~geometry~~



Gravitons are spin 2 particles

- **Not suited for all problems. Works very well for asymptotically flat space-times in context of perturbation theory.**
- **Well suited for gravitational-wave physics from compact astrophysical objects.**



Our Philosophy

"By now the passage of time has taught us not to expect that the strong, weak and electromagnetic interactions can be understood in geometrical terms, and too great an emphasis on geometry can only obscure the deep connections between gravitation and the rest of physics."

Steven Weinberg in preface to his book on gravity

This underlies our approach.

- **Our starting point is gauge theory. (Strong, weak and electromagnetic)**
- ***Double copy* will give gravity direct from gauge theory.**
- **So far mainly useful for perturbative problems around flat space.**

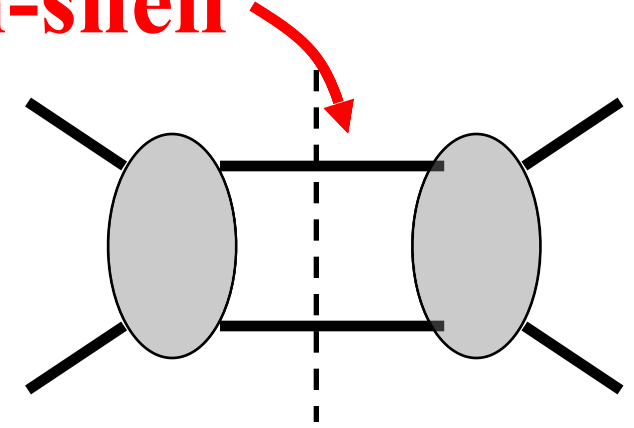
From Tree to Loops: Generalized Unitarity Method

Use tree amplitudes to build higher order (loop) amplitudes.

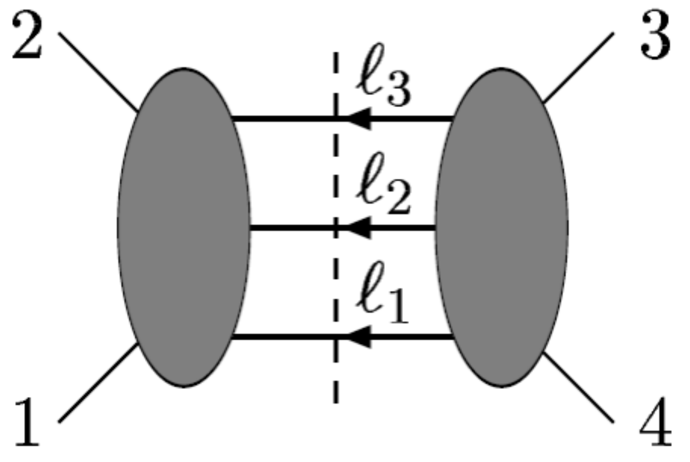
$E^2 = \vec{p}^{\,2} + m^2$ ← on-shell

ZB, Dixon, Dunbar and Kosower (1994)

Two-particle cut:



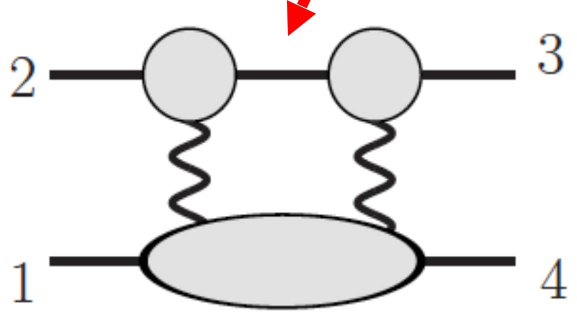
Three-particle cut:

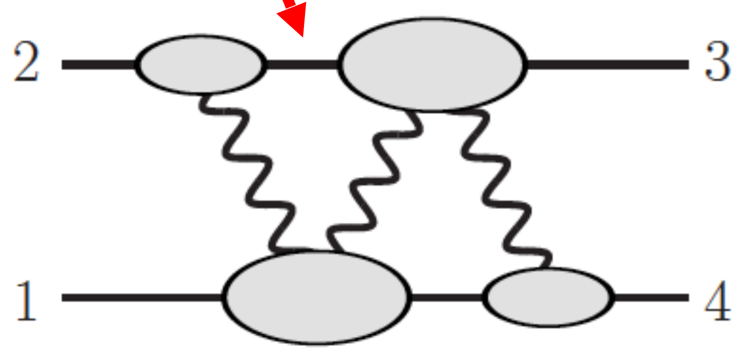


- Systematic assembly of complete loop amplitudes from tree amplitudes.
- Works for any number of particles or loops.

on-shell

Generalized unitarity as a practical tool for loops.





ZB, Dixon and Kosower;
ZB, Morgan;
Britto, Cachazo, Feng;
Ossala,Pittau,Papadopoulos;
Ellis, Kunszt, Melnikov;
Forde; Badger;
ZB, Carrasco, Johansson, Kosower
and many others

Idea used in the "NLO revolution" in QCD collider physics and high loop supergravity calculations.
Are applying it to gravitational wave problem.

Gravity vs Gauge Theory

Consider the Einstein gravity Lagrangian

curvature

Flat-space metric

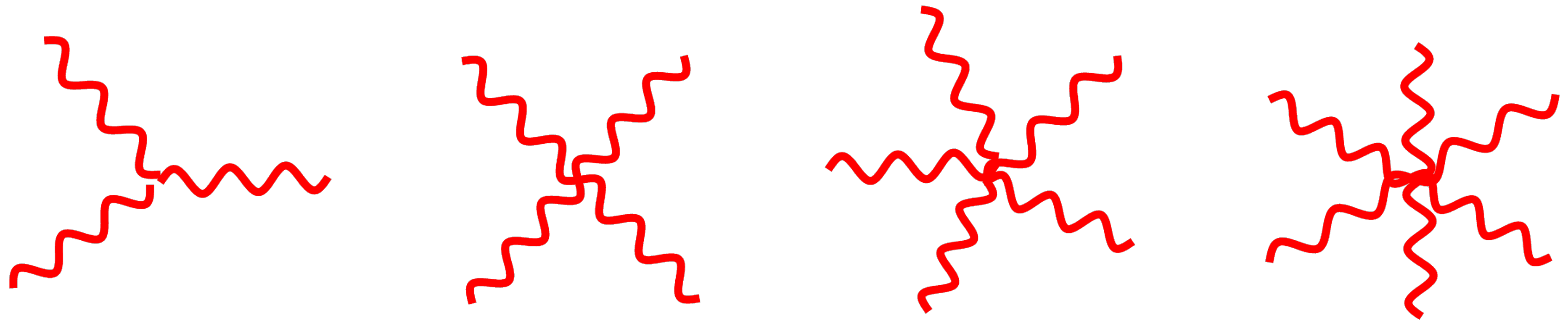
$$L_{\text{gravity}} = \frac{2}{\kappa^2}\sqrt{-g}\,R \qquad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

graviton field

metric

$$\kappa^2 = 32\pi G_{\text{Newton}}$$

Infinite number of complicated interactions

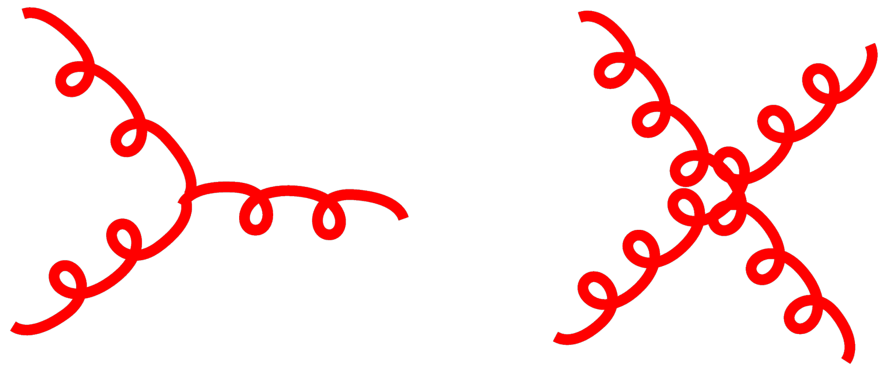


$+ \cdots$

terrible mess

Compare to gauge-theory Lagrangian on which QCD is based

$$L_{\text{YM}} = \frac{1}{g^2}F^2$$



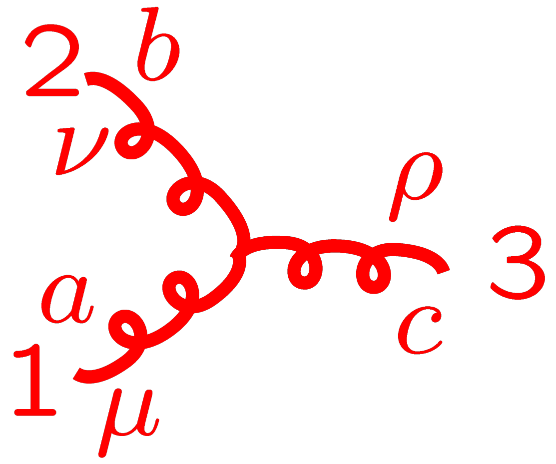
Only three and four point interactions

Gravity seems so much more complicated than gauge theory.

Gauge and gravity theories seem rather different.

Three-Point Interactions

Standard perturbative approach:



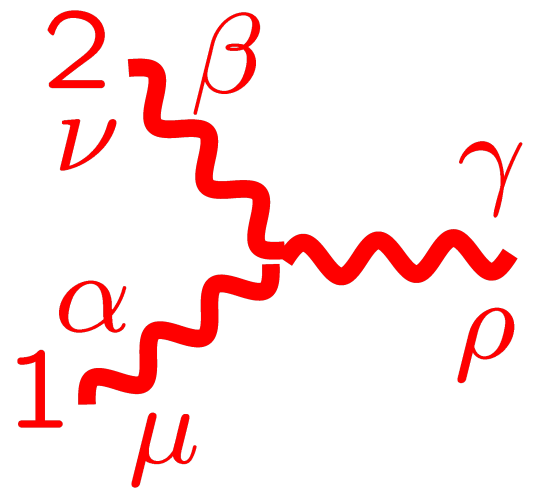
Three-gluon vertex from strong interactions:

$$V_{3\,\mu\nu\sigma}^{abc} = -g f^{abc}(\eta_{\mu\nu}(k_1-k_2)_\rho + \eta_{\nu\rho}(k_1-k_2)_\mu + \eta_{\rho\mu}(k_1-k_2)_\nu)$$

Three-graviton vertex:

$$k_i^2 = E_i^2 - \vec{k}_i^2 \neq 0$$

$$\begin{aligned}
&G_{3\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3) = \\
&\quad \mathrm{sym}[-\tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \tfrac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) + \tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) \\
&\qquad + P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) \\
&\qquad + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) \\
&\qquad + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})]
\end{aligned}$$



About 100 terms in three vertex

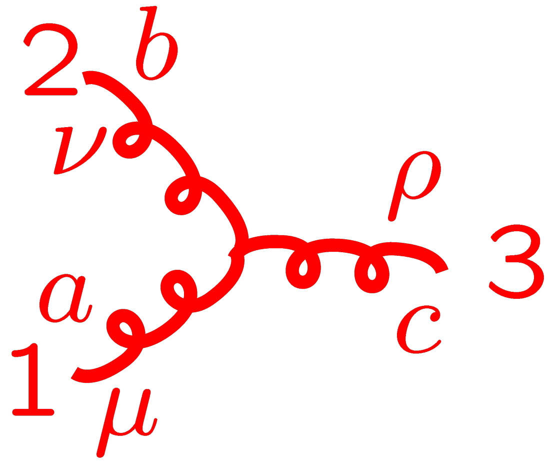
Naïve conclusion: Gravity is a nasty mess.

Simplicity of Gravity Amplitudes

On-shell viewpoint has surprising simplicity.

On-shell **three vertices contains all information:** $k_i^2 = 0$

Yang-Mills gauge theory:

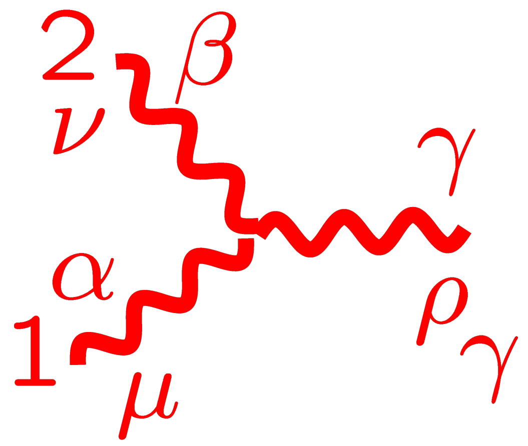


$$-gf^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$

$$\sim gf^{abc}\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

Only consistent vertices with correct dimensions

Einstein gravity:



$$i\kappa(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$
$$\times (\eta_{\alpha\beta}(k_1 - k_2)_\gamma + \text{cyclic})$$
$$\sim i\kappa\left(\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\right)^2$$

"square" of Yang-Mills vertex.

$\kappa^2 = 32\pi G$

Using on-shell methods, BCFW recursion and unitarity method, we can build all tree and loop amplitudes in the theory.

Gravity vs Gauge Theory

Consider the gravity Lagrangian

$$L_{\text{gravity}} = \frac{2}{\kappa^2}\sqrt{-g}\,R$$

$$\kappa^2 = 32\pi\, G_{\text{Newton}}$$

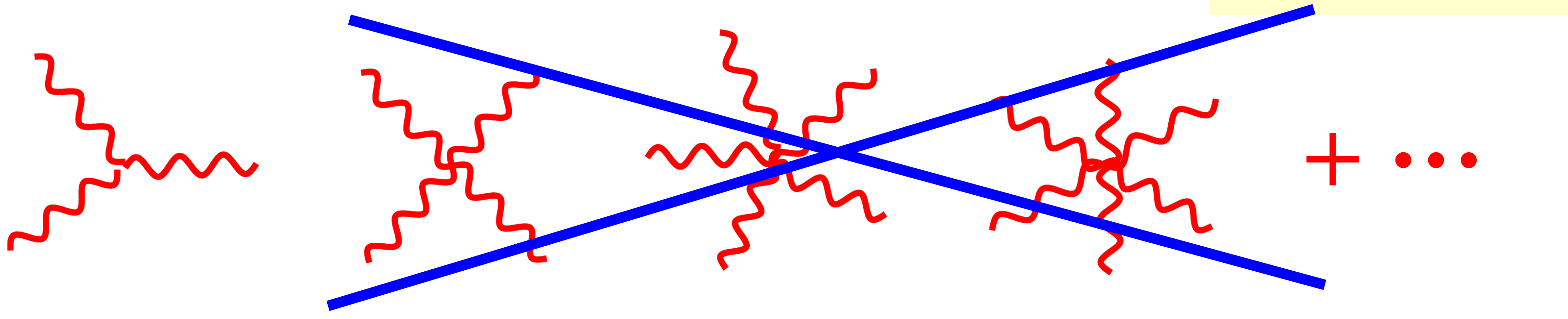
flat metric

graviton field

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

metric

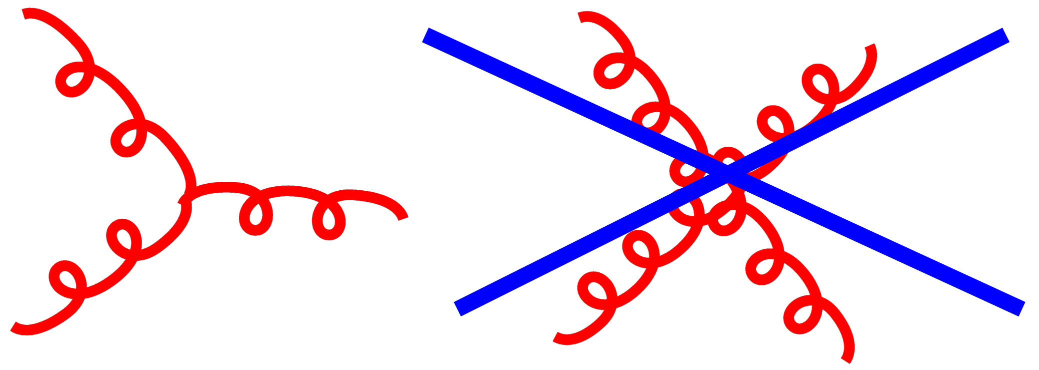
Infinite number of irrelevant interactions!



Simple relation to gauge theory

Compare to Yang-Mills Lagrangian

$$L_{\text{YM}} = \frac{1}{g^2} F^2$$



Only three-point Interactions needed

Gravity seems ~~so much~~ no more complicated than gauge theory.

KLT Relation Between Gravity and Gauge Theory

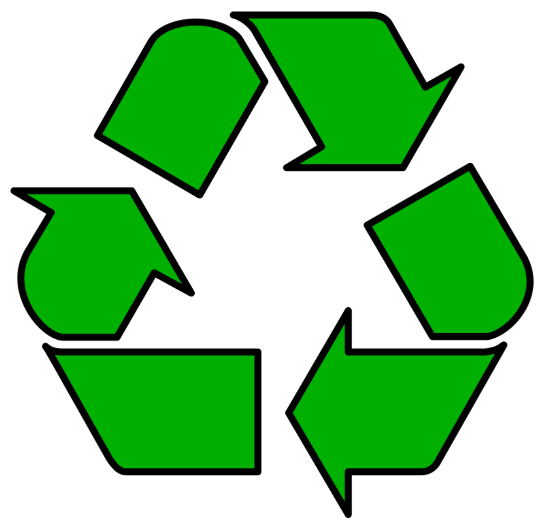
KLT (1985)

Kawai-Lewellen-Tye string relations in low-energy limit:

gravity **gauge-theory color ordered**

$$M_4^{\rm tree}(1,2,3,4) = -i s_{12} A_4^{\rm tree}(1,2,3,4)\, A_4^{\rm tree}(1,2,4,3)\,,$$

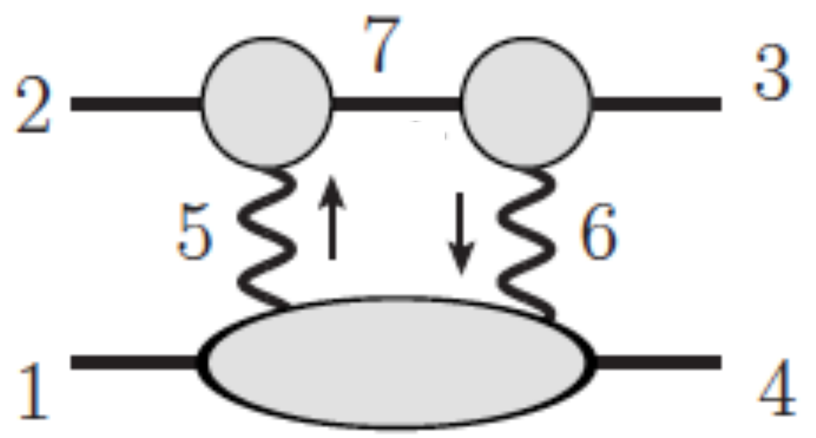
$$M_5^{\rm tree}(1,2,3,4,5) = i s_{12} s_{34} A_5^{\rm tree}(1,2,3,4,5)\, A_5^{\rm tree}(2,1,4,3,5) + i s_{13} s_{24} A_5^{\rm tree}(1,3,2,4,5)\, A_5^{\rm tree}(3,1,4,2,5)$$



Generalizes to explicit all-leg form. ZB, Dixon, Perelstein, Rozowsky

1. **Gravity is derivable from gauge theory. Standard Lagrangian methods offers no hint why this is possible.**
2. **It is very generally applicable.**

Generalized Unitarity Cuts and Double Copy



$$
\begin{aligned}
C_{\rm GR} &= \sum_{h_5,h_6=\pm} M_3^{\rm tree}(3^s, 6^{h_6}, -7^s)\, M_3^{\rm tree}(7^s, -5^{h_5}, 2^s)\, M_4^{\rm tree}(1^s, 5^{-h_5}, -6^{-h_6}, 4^s) \\
&= \sum_{h_5,h_6=\pm} it[A_3^{\rm tree}(3^s, 6^{h_6}, -7^s)\, A_3^{\rm tree}(7^s, -5^{h_5}, 2^s)\, A_4^{\rm tree}(1^s, 5^{-h_5}, -6^{-h_6}, 4^s)] \\
&\qquad \times [A_3^{\rm tree}(3^s, 6^{h_6}, -7^s)\, A_3^{\rm tree}(7^s, -5^{h_5}, 2^s)\, A_4^{\rm tree}(4^s, 5^{-h_5}, -6^{-h_6}, 1^s)]
\end{aligned}
$$

Problem of computing the generalized cuts in gravity is reduced to multiplying and summing gauge-theory tree amplitudes.

Duality Between Color and Kinematics

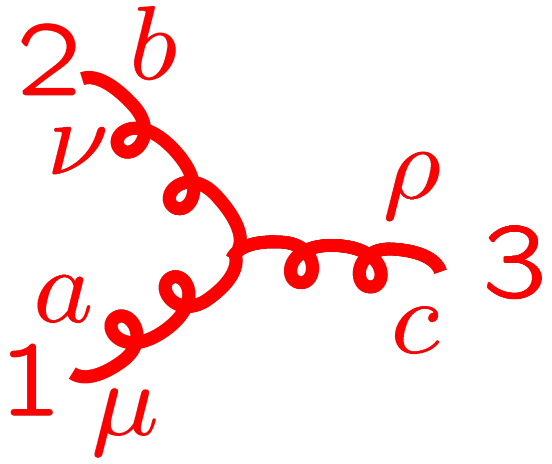
ZB, Carrasco, Johansson

coupling constant

color factor

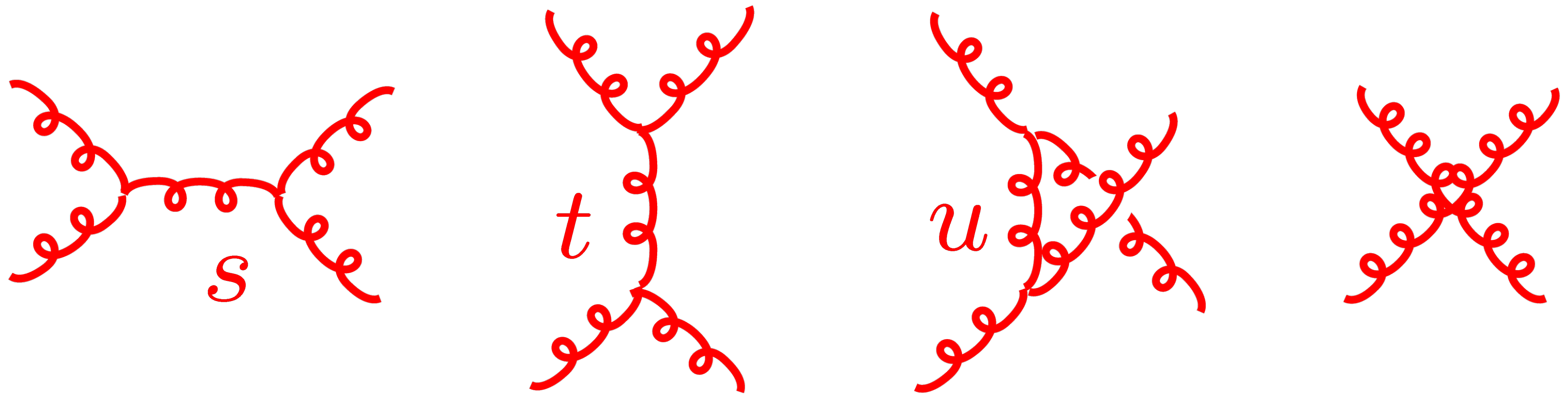
momentum dependent kinematic factor

$$-g f^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$



Color factors based on a Lie algebra: $[T^a, T^b] = i f^{abc} T^c$

Jacobi Identity $f^{a_1 a_2 b} f^{b a_4 a_3} + f^{a_4 a_2 b} f^{b a_3 a_1} + f^{a_4 a_1 b} f^{b a_2 a_3} = 0$



Use 1 = *s/s* = *t/t* = *u/u* to assign 4-point diagram to others.

$s = (k_1 + k_2)^2 \quad t = (k_1 + k_4)^2$

$u = (k_1 + k_3)^2$

$$\mathcal{A}_4^{\text{tree}} = g^2 \left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u} \right)$$

Color factors satisfy Jacobi identity:

Numerator factors satisfy similar identity:

$$\boxed{\begin{aligned} c_u &= c_s - c_t \\ n_u &= n_s - n_t \end{aligned}}$$

Proven at tree level

ZB, Carrasco, Johansson; Kiermaier; Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove; Cachazo, etc

Gravity as a Double copy of Gauge Theory

Kawai, Lewellen, Tye; ZB, Carrasco, Johansson

color factor

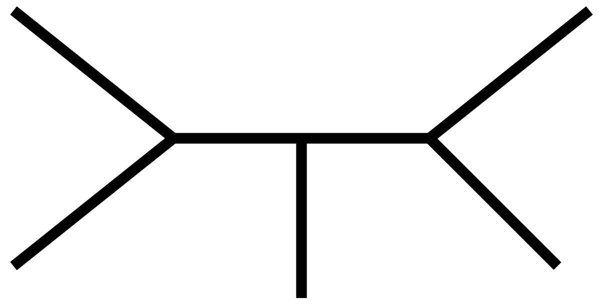
kinematic numerator factor

Feynman propagators

gauge theory (QCD): $$\mathcal{A}_n^{\text{tree}} = i g^{n-2} \sum_i \frac{c_i \, n_i}{D_i}$$

$$c_k = c_i - c_j$$
$$n_k = n_i - n_j$$

$$\boxed{c_i \to n_i}$$



sum over diagrams with only 3 vertices

Einstein gravity: $$\mathcal{M}_n^{\text{tree}} = i \kappa^{n-2} \sum_i \frac{n_i^2}{D_i}$$

$$n_i \sim k_4 \cdot k_5 \, k_2 \cdot \varepsilon_1 \, \varepsilon_2 \cdot \varepsilon_3 \, \varepsilon_4 \cdot \varepsilon_5 + \cdots$$

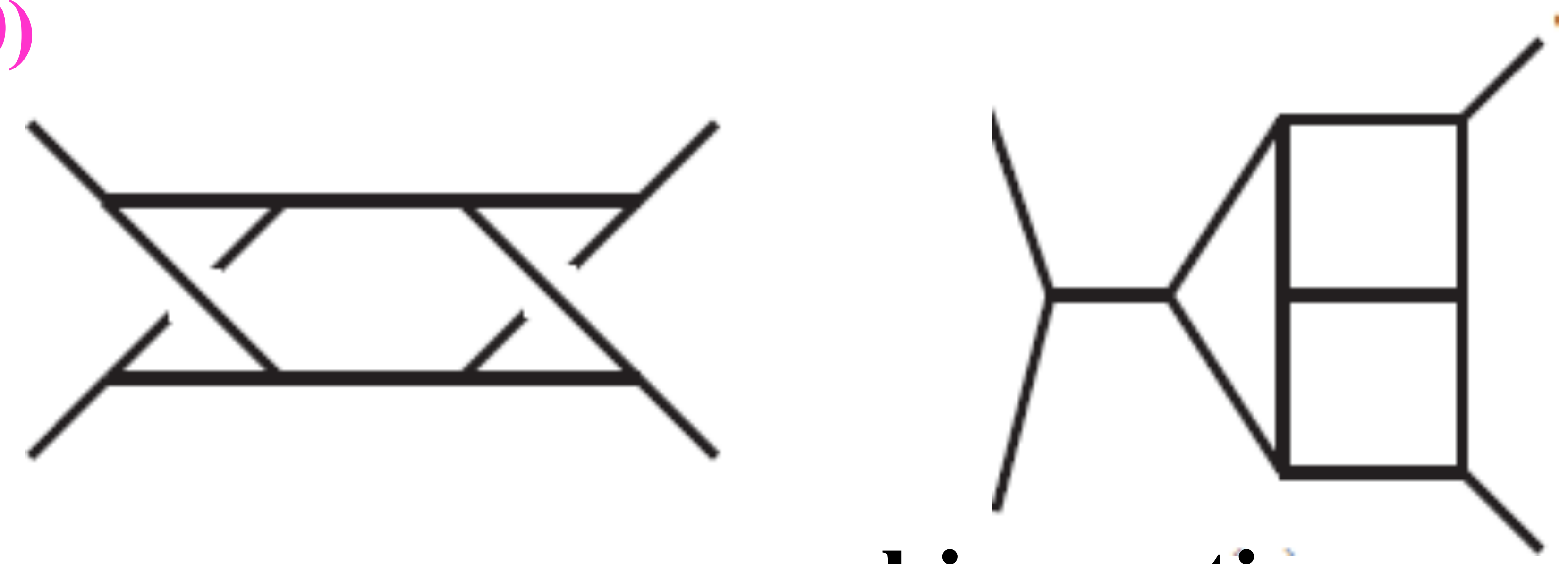
Gravity and gauge theory kinematic numerators are the same!

Same ideas conjectured to hold at loop level.

Cries out for a unified description of gravity with gauge theory, presumably along the lines of string theory.

Loop-Level Generalization

ZB, Carrasco, Johansson (2010)



$$c_i + c_j + c_k = 0$$
$$n_i + n_j + n_k = 0$$

sum is over diagrams

kinematic numerator

color factor

$$\frac{(-i)^L}{g^{n-2+2L}} \mathcal{A}_n^{\text{loop}} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j c_j}{\prod_{\alpha_j} p_{\alpha_j}^2}$$

gauge theory

propagators

$$\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}} \mathcal{M}_n^{\text{loop}} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{\prod_{\alpha_j} p_{\alpha_j}^2}$$

gravity

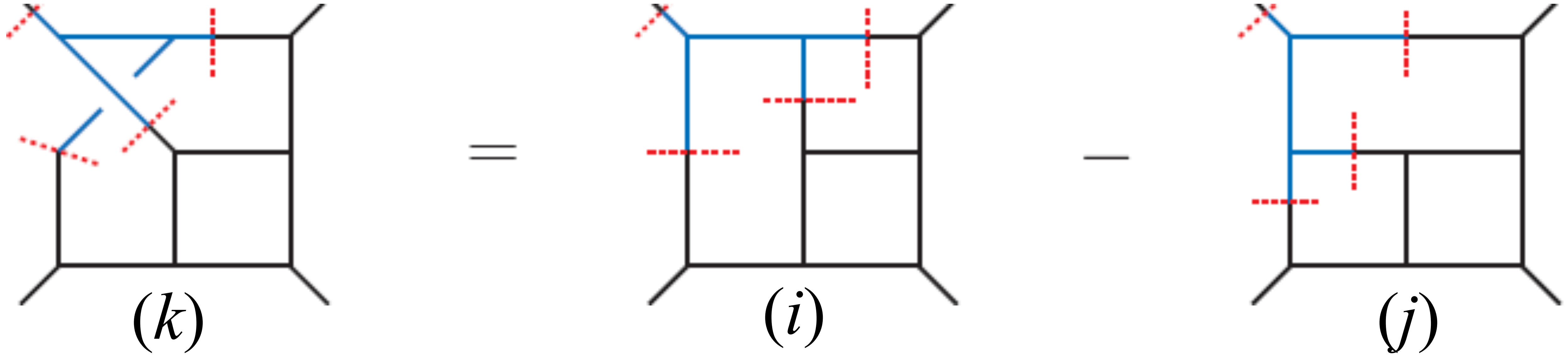
symmetry factor

- Loop-level conjecture is identical to tree-level except for symmetry factors and integration.
- Double copy works if numerator satisfies duality.
- Finding such numerators is nontrivial at high loop orders.

Gravity loop integrands are free!

BCJ

Ideas conjectured to generalize to loops:



color factor

$$c_k = c_i - c_j$$

$$n_k = n_i - n_j$$

kinematic numerator

If you have a set of duality satisfying numerators.
To get:

gauge theory $\longrightarrow$ gravity theory

simply take

color factor $\longrightarrow$ kinematic numerator

$$c_k \longrightarrow n_k$$

Double copy construction

Gravity loop integrands follow from gauge theory!

Generalized Gauge Invariance

ZB, Carrasco and Johansson
ZB, Dennen, Huang, Kiermaier
Tye and Zhang

gauge theory

$$\frac{(-i)^L}{g^{m-2+2L}}\mathcal{A}_m^{\text{loop}} = \sum_j \int \frac{d^{DL}p}{(2\pi)^{DL}} \frac{1}{S_j} \frac{n_j c_j}{\prod_{\alpha_j} p_{\alpha_j}^2}$$

$$n_i \to n_i + \Delta_i \qquad \sum_j \int \frac{d^{DL}p}{(2\pi)^{DL}} \frac{1}{S_j} \frac{\Delta_j c_j}{\prod_{\alpha_j} p_{\alpha_j}^2} = 0$$

Cancellations at integrand level

Above is just a definition of generalized gauge invariance.
Includes ordinary gauge invariance.

gravity

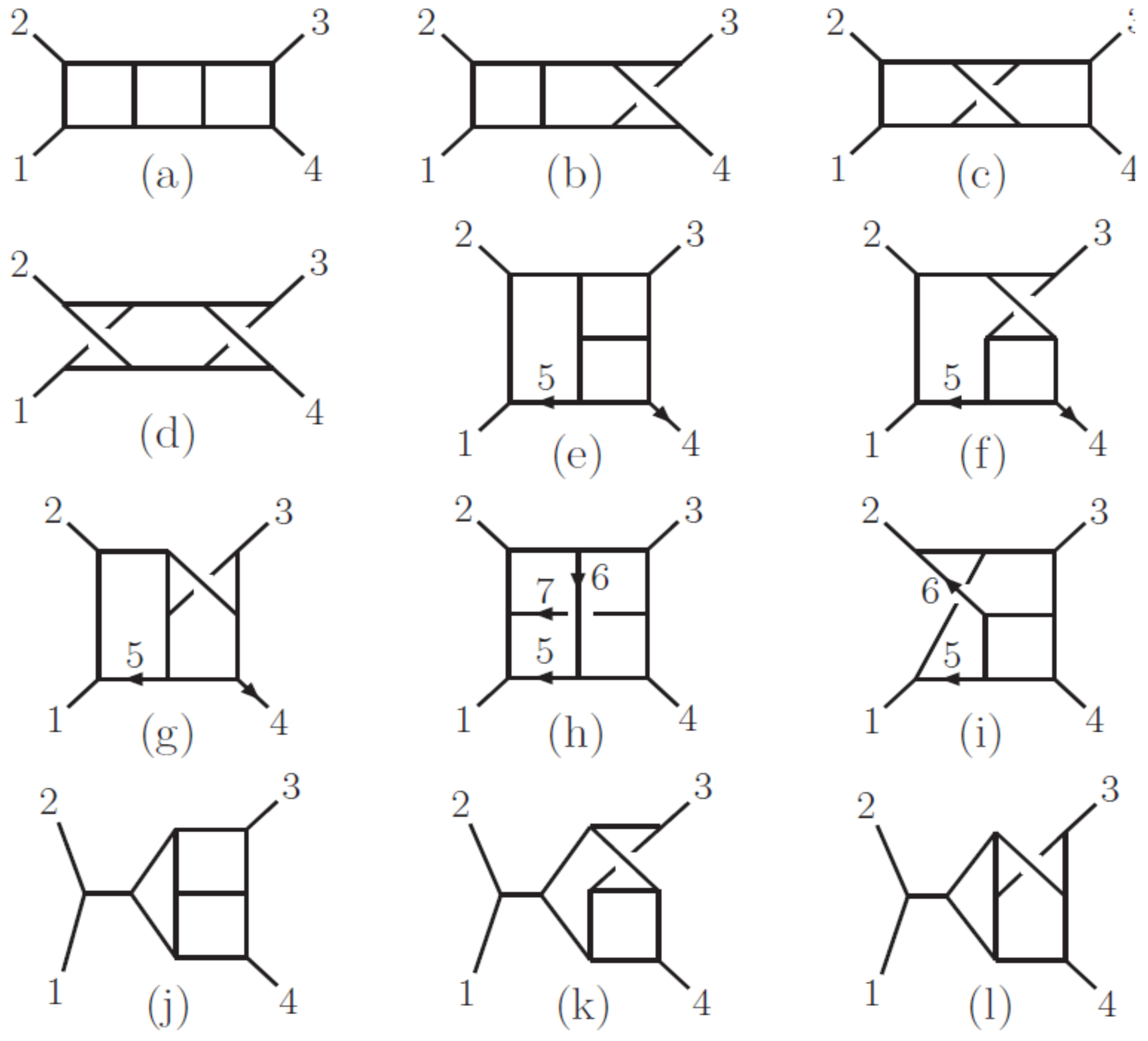
$$\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}}\mathcal{M}_n^{\text{loop}} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{\prod_{\alpha_j} p_{\alpha_j}^2}$$

$$n_i \to n_i + \Delta_i \qquad \sum_j \int \frac{d^{DL}p}{(2\pi)^{DL}} \frac{1}{S_j} \frac{\Delta_j \tilde{n}_j}{\prod_{\alpha_j} p_{\alpha_j}^2} = 0$$

- **Gravity inherits generalized gauge invariance from gauge theory!**
- **Double copy works even if only one of the two copies has duality manifest!**

Three loop $\mathcal{N} = 4$ sYM integrand in BCJ Form

$\mathcal{N} = 8$ sugra : ($\mathcal{N} = 4$ sYM) x ($\mathcal{N} = 4$ YM)



- **$\mathcal{N} = 4$ sYM numerators**
- **$\mathcal{N} = 8$ sugra numerator obtained by squaring them.**

Integral $I^{(x)}$	$\mathcal{N} = 4$ Super-Yang-Mills ($\sqrt{\mathcal{N} = 8 \text{ supergravity}}$) numerator
(a)–(d)	s^2
(e)–(g)	$\left(s\,(-\tau_{35} + \tau_{45} + t) - t\,(\tau_{25} + \tau_{45}) + u\,(\tau_{25} + \tau_{35}) - s^2 \right)/3$
(h)	$\left(s\,(2\tau_{15} - \tau_{16} + 2\tau_{26} - \tau_{27} + 2\tau_{35} + \tau_{36} + \tau_{37} - u) \right.$ $\left. + t\,(\tau_{16} + \tau_{26} - \tau_{37} + 2\tau_{36} - 2\tau_{15} - 2\tau_{27} - 2\tau_{35} - 3\tau_{17}) + s^2 \right)/3$
(i)	$\left(s\,(-\tau_{25} - \tau_{26} - \tau_{35} + \tau_{36} + \tau_{45} + 2t) \right.$ $\left. + t\,(\tau_{26} + \tau_{35} + 2\tau_{36} + 2\tau_{45} + 3\tau_{46}) + u\,\tau_{25} + s^2 \right)/3$
(j)-(l)	$s(t-u)/3$

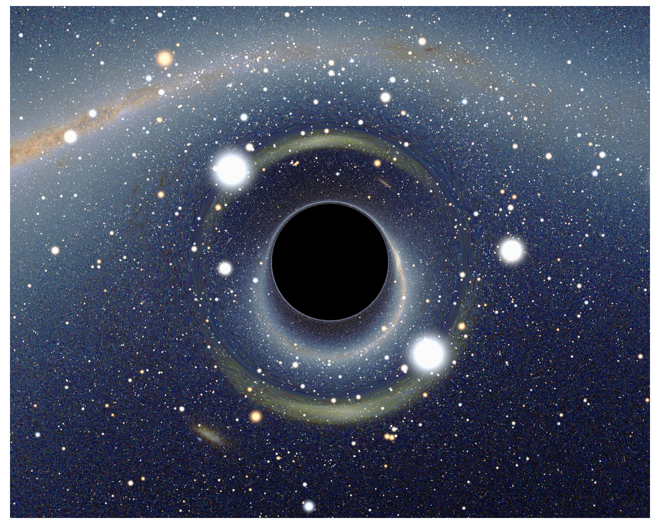
Overall factor of $st\,A_4^{\text{tree}}$

$$s = (p_1 + p_2)^2$$
$$t = (p_2 + p_3)^2$$
$$\tau_{ij} = 2 p_l \cdot l_i$$

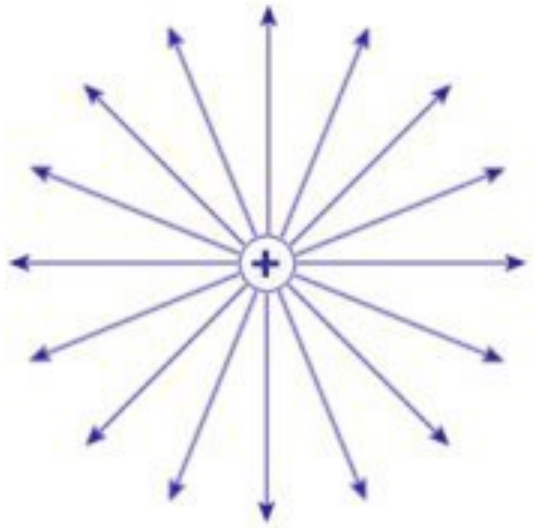
Applications to Black Hole Physics

Wouldn't it be really neat if every classical solution in gravity could be mapped to a double copy of classical solutions?

Where to start? **Obviously the coolest place possible: black holes.**



black hole



point charge

Monteiro, O'Connell and White

Special coordinates: Kerr-Schild coordinates:

Schwarzschild black hole

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi k_\mu k_\nu \qquad \phi(r) = \frac{2m}{r}$$

k is null

Coulomb point charge

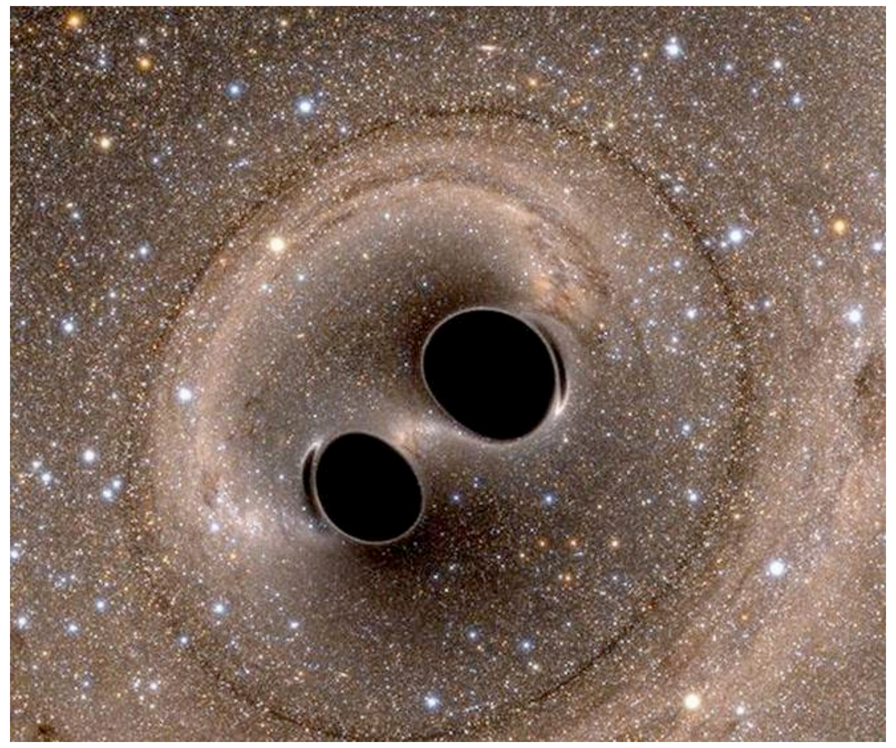
$$A_\mu = \phi k_\mu \qquad \phi(r) = \frac{Q}{r}$$

Schwarzschild ~ (Coulomb)²

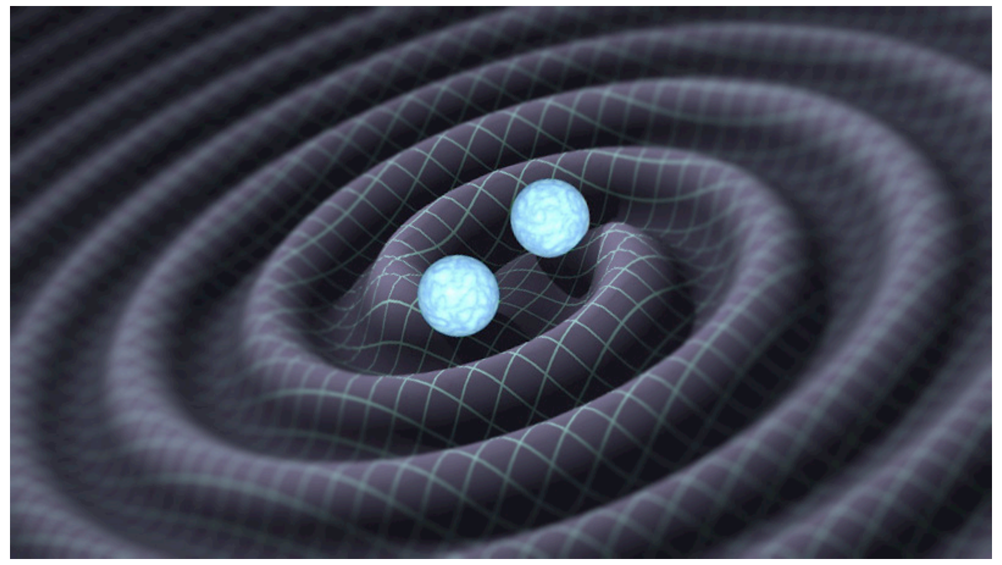
Double Copy for Classical Solutions

Goal is to formulate gravity solutions directly in terms of gauge theory

Variety of special cases:



- **Schwarzschild and Kerr black holes.**
- **Solutions with cosmological constant.**
- **Radiation from accelerating black hole.**
- **Maximally symmetric space times.**
- **Plane wave background.**
- **Gravitational radiation.**



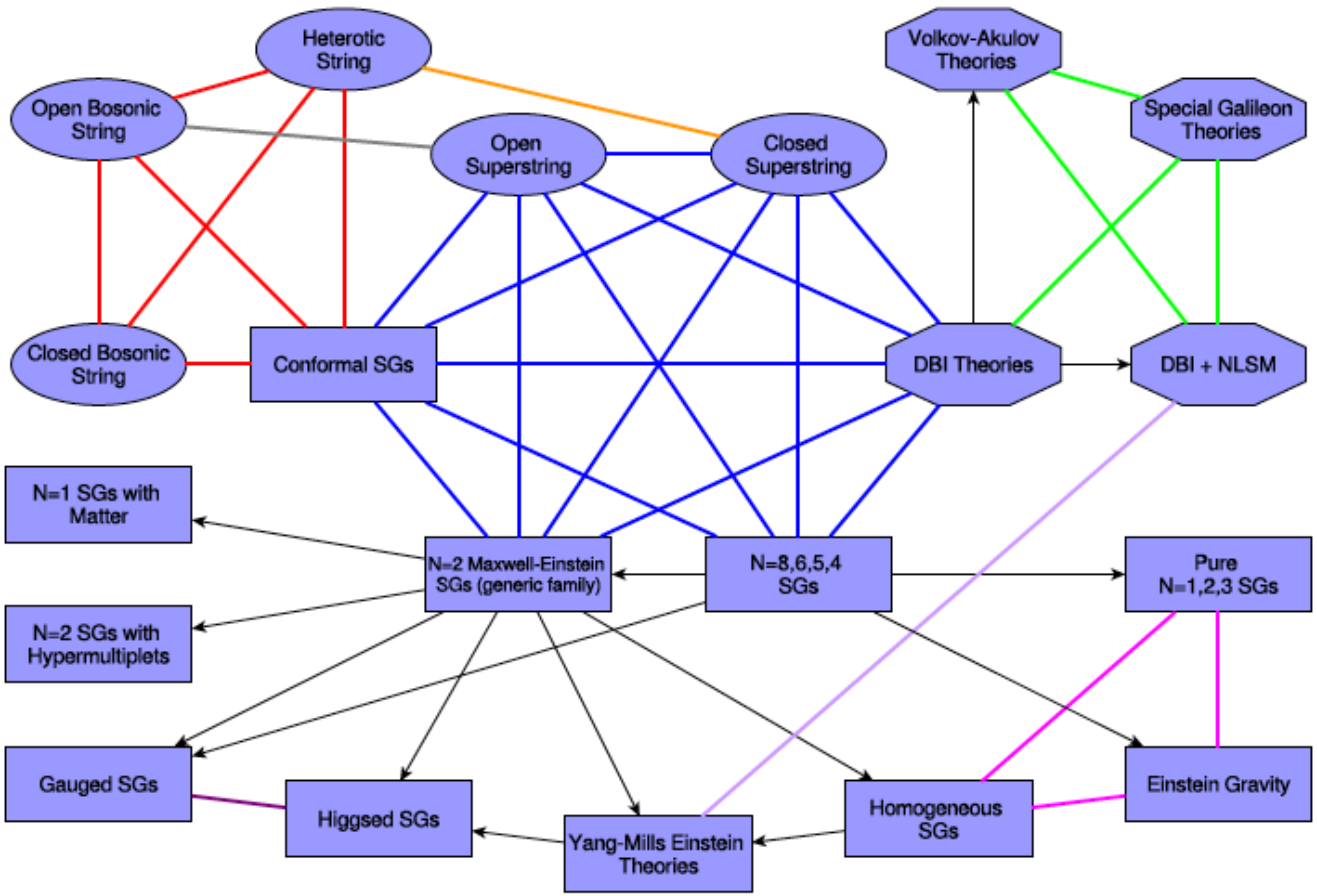
Monteiro, O'Connell and White;
Luna, Monteiro, O'Connell and White;
Luna, Monteiro, Nicholsen, O'Connell and White;
Ridgway and Wise; Carrillo González, Penco, Trodden;
Adamo, Casali, Mason, Nekovar;

Goldberger and Ridgway; Chen;
Luna, Monteiro, Nicholson, Ochirov;
Bjerrum-Bohr, Donoghue, Vanhove;
O'Connell, Westerberg, White; Kosower, Maybee, O'Connell; Adamo, Casali, Mason, Nekovar
etc

Still no general understanding.
But plenty of examples.

Web of Theories

ZB, Carrasco, Chiodaroli, Johansson, Roiban arXiv:1909.01358, Section 5.



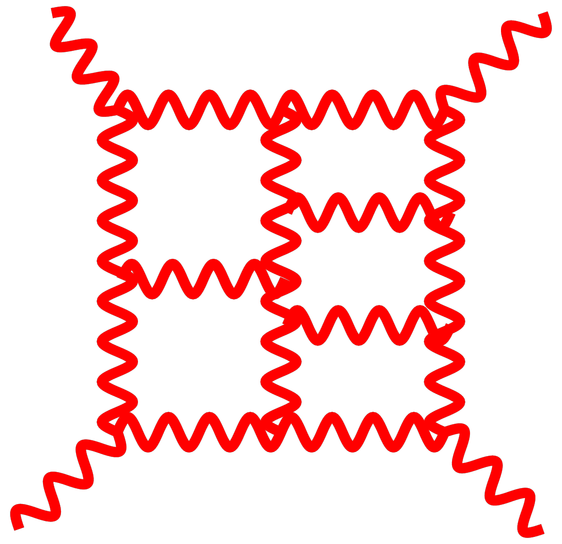
Double copy links various theories through their component theories.

Summary

In a very precise sense:

Gravity ~ (gauge theory) x (gauge theory)

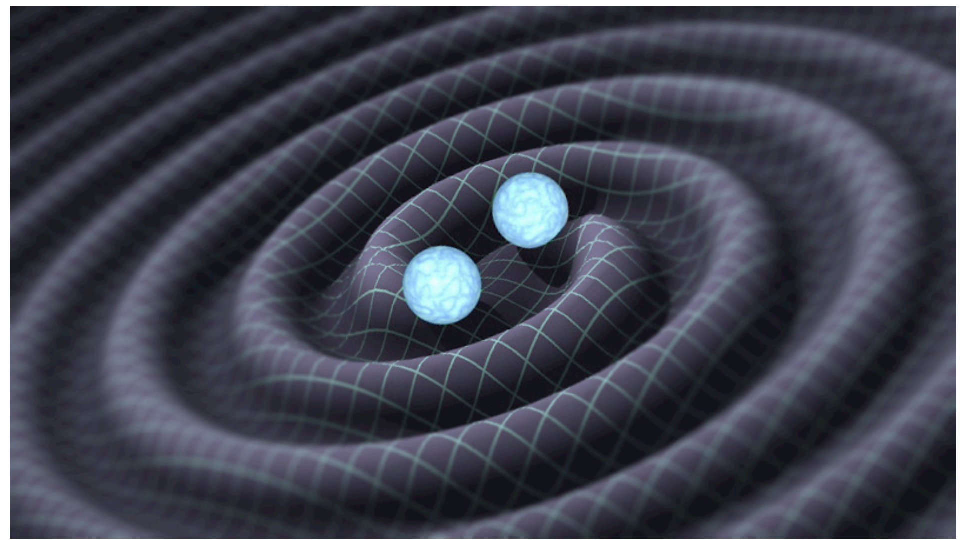
Examples of Applications:



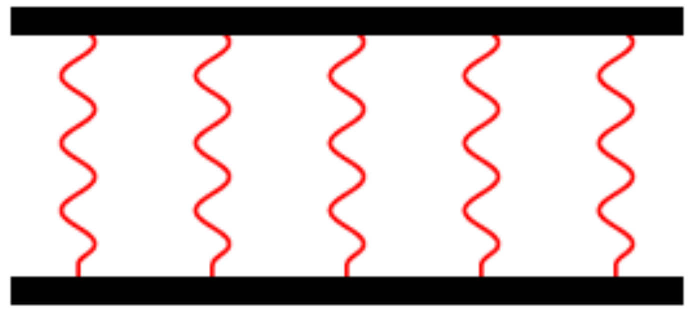
- **4,5 loop supergravity to study nonrenormalizability of gravity theories.**

ZB, Carrasco, Dixon, Johansson, Roiban; ZB, Davies, Dennen; ZB, Carrasco, Chen, Edison, Johansson, Roiban, Parra-Martinez, Zeng

- **$G^3 - G^5$ corrections to Newton's potential from GR (all orders in velocity).**



ZB, Cheung, Roiban, Shen, Solon, Zeng (2019)
ZB, Parra-Martinez, Roiban, Ruf. Shen, Solon, Zeng (2021)
ZB, Herrmann, Roiban, Ruf, Smirnov, Smirnov (2024)



Understanding UV of Gravity

Quantum Gravity

Often repeated statement:

"Einstein's theory of General Relativity is incompatible with quantum mechanics."

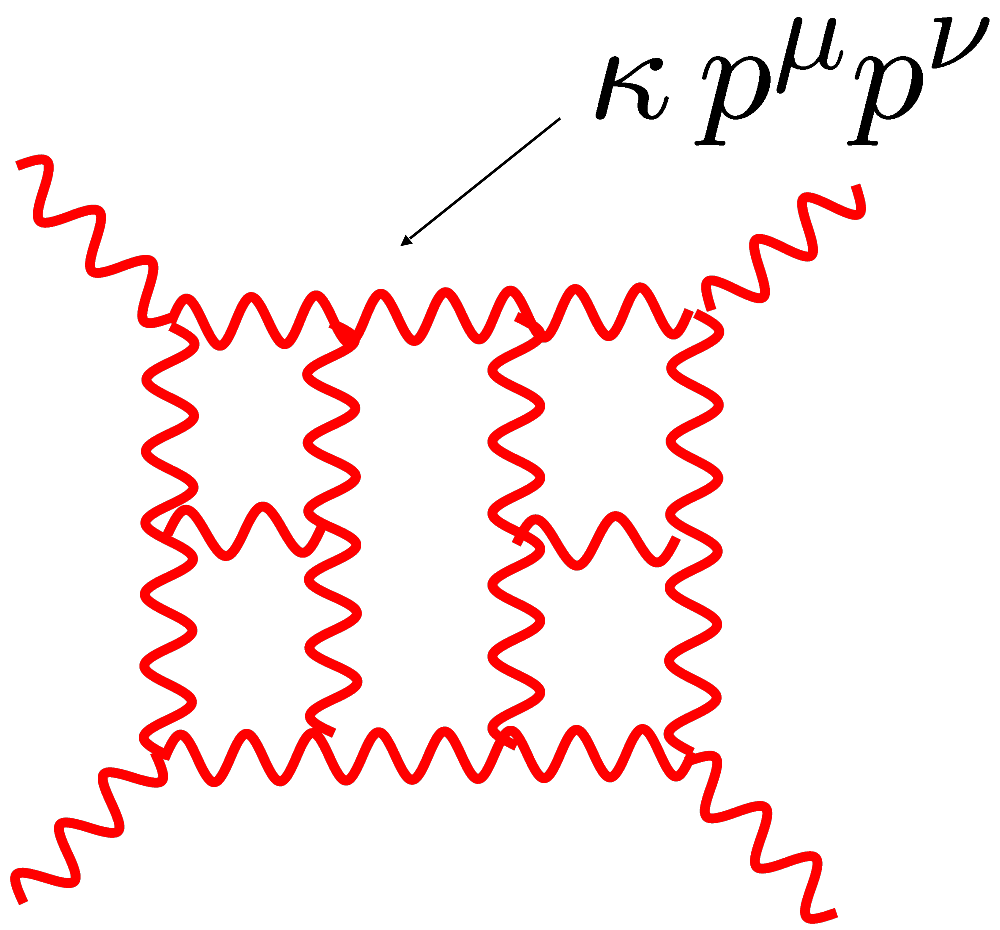
To a large extent this is based on another often repeated statement:

"All point-like quantum theories of gravity are divergent and non-renormalizable."

Where do these statements come from and are they true?

UV Behavior of Gravity?

$$\kappa = \sqrt{32\pi G_N}$$ ⟵ Dimensionful coupling



Gravity: $$\int \prod_{i=1}^{L} \frac{d^D p_i}{(2\pi)^D} \frac{\cdots \kappa p_j^\mu p_j^\nu \cdots}{\text{propagators}}$$

Gauge theory: $$\int \prod_{i=1}^{L} \frac{d^D p_i}{(2\pi)^D} \frac{\cdots g p_j^\nu \cdots}{\text{propagators}}$$

- **Extra powers of loop momenta in numerator means integrals are badly behaved in the UV and must diverge at some loop order.**
- **Much more sophisticated power counting in supersymmetric theories but this is basic idea.**

- **With more supersymmetry expect better UV properties.**
- **Need to worry about "hidden cancellations".**
- **$N = 8$ supergravity *best* theory to study.**

Test case: $N = 8$ Supergravity

The best theories to look at are supersymmetric theories.

Supersymmetry relates bosons (forces) and fermions (matter)

We first consider $N = 8$ supergravity.

Einstein gravity + 254 other physical states

Reasons to focus on $N \geq 4$ supergravity:
- **With more supersymmetry expect better UV properties.**
- **High symmetry implies technical simplicity.**

In the late 70's and early 80's supergravity was seen as the primary means for unifying gravity with other forces.

Ferrara, Freedman, van Nieuwenhuizen

Opinions from the 80's

If certain patterns that emerge should persist in the higher orders of perturbation theory, then … $N = 8$ supergravity in four dimensions would have ultraviolet divergences starting at three loops. Green, Schwarz, Brink, (1982)

It is therefore very likely that all supergravity theories will diverge at three loops in four dimensions… The final word on these issues may have to await further explicit calculations.

Marcus, Sagnotti (1985)

Note the lack of certainty and caution in these opinions.

$N = 8$ supergravity: Where is First $D = 4$ UV Divergence?

3 loops $N = 8$	Green, Schwarz, Brink (1982); Howe and Stelle (1989); Marcus and Sagnotti (1985)	✗
5 loops $N = 8$	Bern, Dixon, Dunbar, Perelstein, Rozowsky (1998); Howe and Stelle (2003,2009)	✗
6 loops $N = 8$	Howe and Stelle (2003)	✗
7 loops $N = 8$	Grisaru and Siegel (1982); Bossard, Howe, Stelle (2009);Vanhove; Björnsson, Green (2010); Kiermaier, Elvang, Freedman(2010); Ramond, Kallosh (2010); Biesert et al (2010); Bossard, Howe, Stelle, Vanhove (2011)	?
3 loops $N = 4$	Bossard, Howe, Stelle, Vanhove (2011)	✗
4 loops $N = 5$	Bossard, Howe, Stelle, Vanhove (2011)	✗
4 loops $N = 4$	Vanhove and Tourkine (2012)	✓
9 loops $N = 8$	Berkovits, Green, Russo, Vanhove (2009)	✗

"shut up and calculate"

ZB, Kosower, Carrasco, Dixon, Johansson, Roiban; ZB, Davies, Dennen, A. Smirnov, V. Smirnov; series of calculations.

This is what we are most interested in.

Weird structure. Anomaly-like behavior of divergence.

Retracted, but perhaps to be unretracted.

- **Track record of predictions from standard symmetries not great.**
- **Conventional wisdom holds that it will diverge sooner or later.**

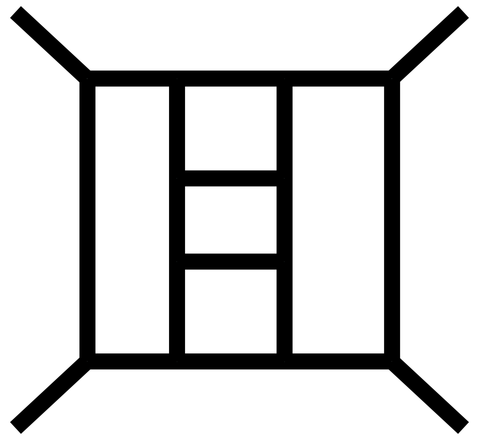
Supersymmetry and Ultraviolet Divergences

Bossard, Howe, Stelle; Elvang, Freedman, Kiermaier; Green, Russo, Vanhove ; Green and Björnsson ; Bossard , Hillmann and Nicolai; Ramond and Kallosh; Broedel and Dixon; Elvang and Kiermaier; Beisert, Elvang, Freedman, Kiermaier, Morales, Stieberger; Bossard, Howe, Stelle, Vanhove, etc

Poor UV behavior in N = 8 supergravity unless new types of cancellations between diagrams exist that are "not consequences of supersymmetry in any conventional sense"

Bjornsson and Green

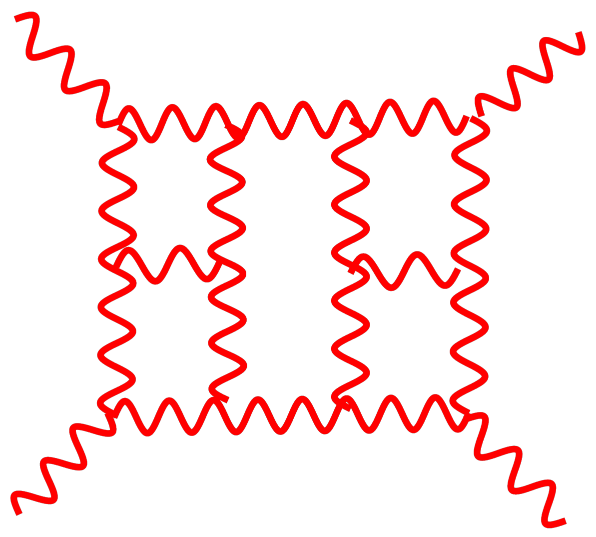
- **$N = 4$ sugra should diverge at 3 loops in $D = 4$.**
- **$N = 5$ sugra should diverge at 4 loops in $D = 4$.**
- **Half maximal sugra diverges at 2 loops in $D = 5$.**
- **$N = 8$ sugra should diverge at 5 loops in $D = 24/5$.**
- **$N = 8$ sugra should diverge at 7 loops in $D = 4$.**



Consensus agreement from all power-counting methods.

Scorecard on Symmetry Predictions

- $\mathcal{N} = 4$ sugra should diverge at 3 loops in $D = 4$. ✗
- $\mathcal{N} = 5$ sugra should diverge at 4 loops in $D = 4$. ✗
- Half maximal sugra diverges at 2 loops in $D = 5$. ✗
- $\mathcal{N} = 8$ sugra should diverge at 5 loops in $D = 24/5$. ✓
- $\mathcal{N} = 8$ sugra should diverge at 7 loops in $D = 4$. ?
- $\mathcal{N} = 5$ sugra should diverge at 5 loops in D = 4. ?



key questions

ZB, Davies, Dennen (2012, 2014); ZB, Davies, Dennen, Huang(2012)

ZB, Carrasco, Chen, Edison, Johansson, Roiban, Parra-Martinez, Zeng (2018)

- UV cancellation of $\mathcal{N} = 5$ supergravity at 4 loops in $D = 4$ definite mystery. Standard symmetry arguments do appear power enough.

Freedman, Kallosh and Yamada (2018)

What is the difference between $\mathcal{N} = 5$ and $\mathcal{N} = 8$?
$D = 4$ has extra cancellations.

Edison, Herrmann, Parra-Martinez, Trnka (2019)

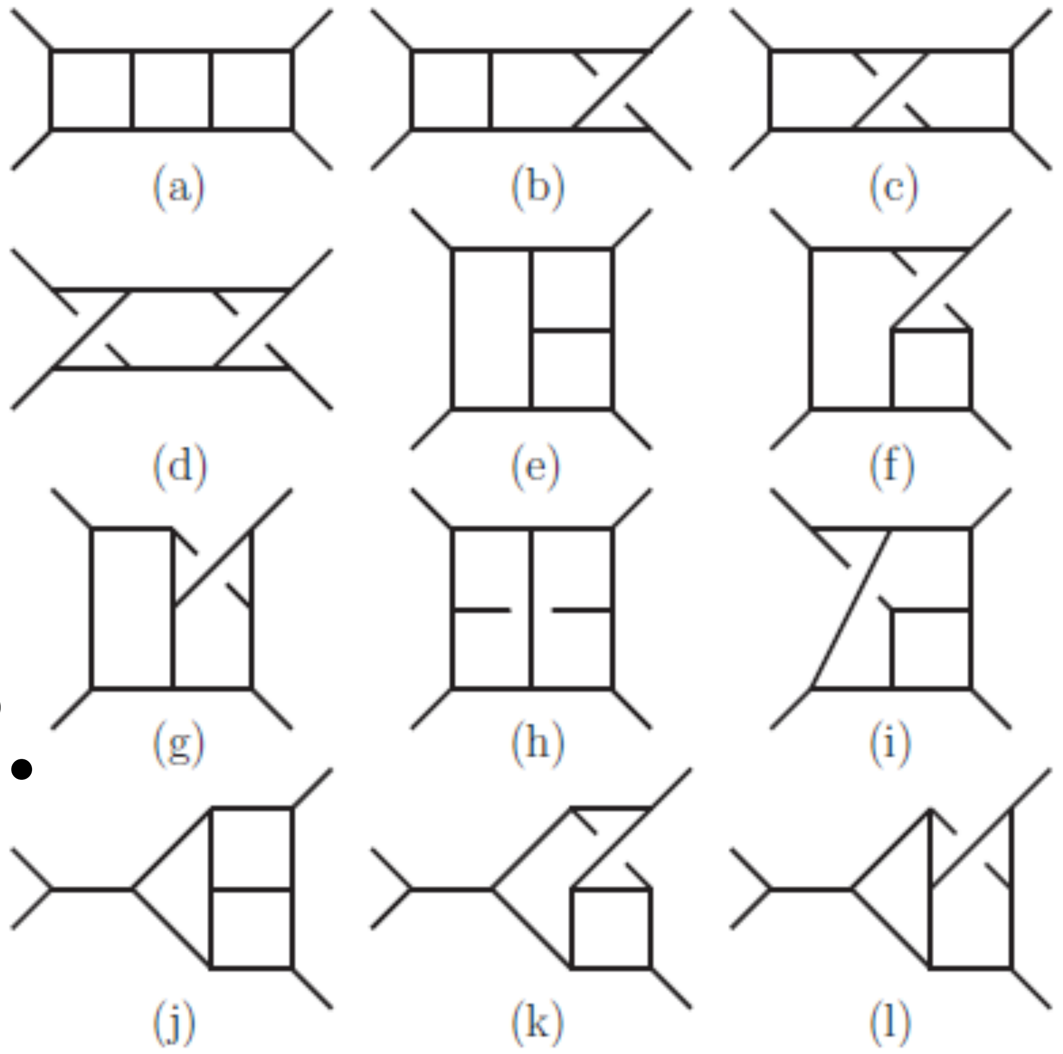
Goal is to provide definitive answers. Key target $\mathcal{N} = 5$ supergravity at 5 loops.

ZB, Chen, Edison, Gopalka, Jones, Herrmann, Roiban, Ruf (ongoing)

Enhanced UV Cancellations

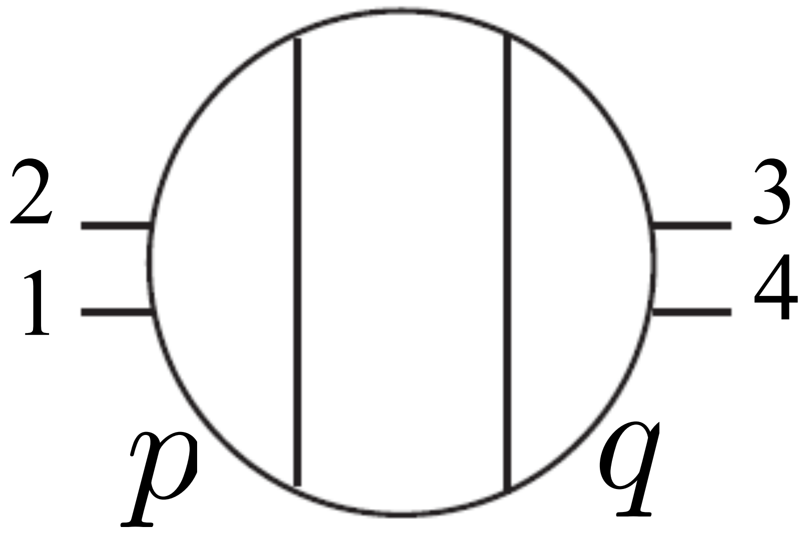
ZB, Davies, Dennen (2014)

Suppose diagrams in *all* possible Lorentz covariant representations are UV divergent, but the amplitude is well behaved.



- **By definition this is an "enhanced cancellation".**
- **Not the way gauge theory works.**
- **Normal formulations don't display these.**

$N = 4$ sugra



already log divergent

$N = 4$ sugra: pure YM x $N = 4$ sYM

$$n_i \sim s^3 t A_4^{\text{tree}} \, (p \cdot q)^2 \, \varepsilon_1 \cdot p \, \varepsilon_2 \cdot p \, \varepsilon_3 \cdot q \, \varepsilon_4 \cdot q + \ldots$$

This diagram is log divergent

3 loop UV finiteness of $N = 4$ supergravity proves existence of "enhanced cancellation" in supergravity theories.

$\mathcal{N} = 5$ Supergravity Four-Loop Enhanced Cancellations

ZB, Davies and Dennen

Industrial strength software needed: FIRE and special purpose C++

Crucial help from (Smirnov)^2

$\mathcal{N} = 5$ sugra: ($\mathcal{N} = 4$ sYM) x ($\mathcal{N} = 1$ sYM)

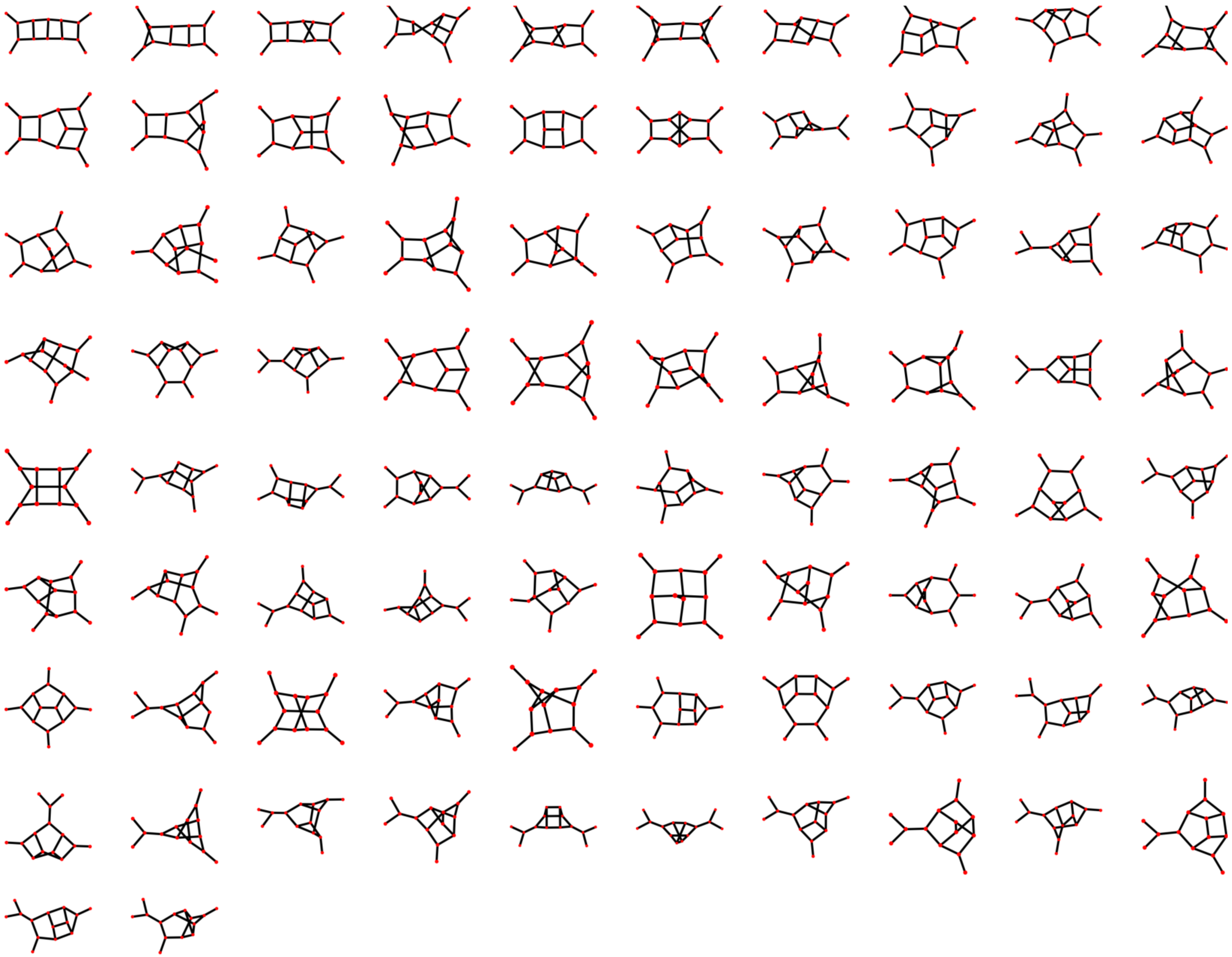
$\mathcal{N} = 4$ sYM **$\mathcal{N} = 1$ sYM**

Certain diagrams necessarily UV divergent.

Enhanced cancellation: $\mathcal{N} = 5$ supergravity at 4 loops UV finite despite the fact that individual pieces are divergent.

82 nonvanishing numerators in BCJ representation

ZB, Carrasco, Dixon, Johansson, Roiban ($\mathcal{N} = 4$ sYM)



$N = 5$ Supergravity Divergences at Four Loops

ZB, Davies and Dennen

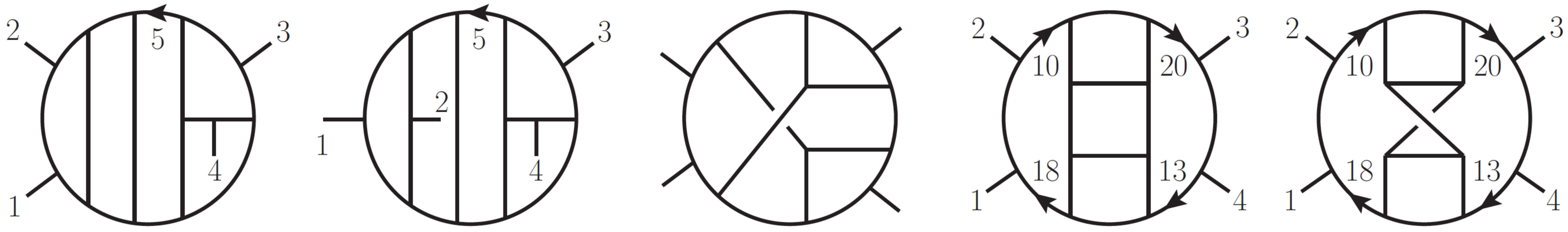
graphs	(divergence) $\times\, u/(-i/(4\pi)^8\langle 12\rangle^2[34]^2 st A^{\text{tree}}(\frac{\kappa}{2})^{10})$
1–30	$\frac{1}{\epsilon^4}\Big[\frac{7358585}{7962624}s^2+\frac{2561447}{2654208}st-\frac{872683}{1990656}t^2\Big]+\frac{1}{\epsilon^3}\Big[\frac{75972559}{35389440}s^2+\frac{240984061}{26542080}st+\frac{1302037}{1310720}t^2\Big] +\frac{1}{\epsilon^2}\Big[\zeta_3\big(-\frac{369234283}{11059200}s^2-\frac{257792411}{4915200}st-\frac{101847769}{14745600}t^2\big)+\zeta_2\big(\frac{7358585}{3981312}s^2+\frac{2561447}{1327104}st-\frac{872683}{995328}t^2\big) -\mathrm{S2}\big(\frac{1223621}{49152}s^2+\frac{46816475}{442368}st+\frac{2639903}{221184}t^2\big)+\frac{206093335871}{11466178560}s^2+\frac{320983191023}{3822059520}st+\frac{53309416589}{2866544640}t^2\Big] +\frac{1}{\epsilon}\Big[\zeta_5\big(-\frac{84777347}{368640}s^2+\frac{382194721}{1474560}st+\frac{417476581}{1474560}t^2\big)-\zeta_4\big(\frac{3062401}{2457600}s^2+\frac{3881051}{3276800}st-\frac{112081813}{29491200}t^2\big) +\zeta_3\big(\frac{28162691399797}{53747712000}s^2+\frac{19354492750651}{35831808000}st-\frac{22092683352811}{107495424000}t^2\big)-\zeta_2\big(\frac{70861961}{17694720}s^2+\frac{227180689}{13271040}st +\frac{105727243}{53084160}t^2\big)+\mathrm{T1ep}\big(-\frac{1223621}{663552}s^2-\frac{46816475}{5971968}st-\frac{2639903}{2985984}t^2\big)-\mathrm{S2}\big(\frac{11916028151}{5898240}s^2 +\frac{72637733971}{13271040}st+\frac{17223563447}{53084160}t^2\big)+\mathrm{D6}\big(-\frac{9001177}{552960}s^2-\frac{264491}{10240}st-\frac{2610157}{552960}t^2\big) +\frac{110945914744727}{1146617856000}s^2+\frac{16989492195991}{127401984000}st-\frac{21362122998269}{573308928000}t^2\Big]$
31–60	$\frac{1}{\epsilon^4}\Big[-\frac{5502451}{2654208}s^2-\frac{3675877}{884736}st+\frac{11269}{497664}t^2\Big]+\frac{1}{\epsilon^3}\Big[\frac{38102993}{26542080}s^2-\frac{291607201}{106168320}st-\frac{565798829}{318504960}t^2\Big] +\frac{1}{\epsilon^2}\Big[\zeta_3\big(\frac{108955183}{2211840}s^2+\frac{653019571}{8847360}st+\frac{9453043}{1769472}t^2\big)+\zeta_2\big(-\frac{5502451}{1327104}s^2-\frac{3675877}{442368}st+\frac{11269}{248832}t^2\big) +\mathrm{S2}\big(\frac{16797481}{1327104}s^2+\frac{1172969}{16384}st+\frac{978427}{82944}t^2\big)-\frac{304243754383}{19110297600}s^2-\frac{2032063711381}{19110297600}st-\frac{257798086613}{7166361600}t^2\Big] +\frac{1}{\epsilon}\Big[\zeta_5\big(\frac{33327659}{122880}s^2+\frac{13276219}{24576}st+\frac{22251887}{184320}t^2\big)+\zeta_4\big(\frac{12299887}{1474560}s^2+\frac{258056147}{5898240}st+\frac{46913759}{5898240}t^2\big) +\zeta_3\big(-\frac{26846001990157}{42998169600}s^2-\frac{337106527201}{265420800}st-\frac{5298324906787}{42998169600}t^2\big)+\zeta_2\big(\frac{282283789}{39813120}s^2+\frac{975199319}{53084160}st +\frac{60394451}{159252480}t^2\big)+\mathrm{T1ep}\big(\frac{16797481}{17915904}s^2+\frac{1172969}{221184}st+\frac{978427}{1119744}t^2\big)+\mathrm{S2}\big(\frac{10516980893}{4976640}s^2 +\frac{389045625329}{53084160}st+\frac{216032337589}{159252480}t^2\big)+\mathrm{D6}\big(\frac{503413}{23040}s^2+\frac{12342607}{552960}st+\frac{3661}{184320}t^2\big) -\frac{166777358259461}{1146617856000}s^2-\frac{565137511429117}{1146617856000}st-\frac{21629055712141}{191102976000}t^2\Big]$
61–82	$\frac{1}{\epsilon^4}\Big[\frac{285899}{248832}s^2+\frac{1058273}{331776}st+\frac{275869}{663552}t^2\Big]+\frac{1}{\epsilon^3}\Big[-\frac{380329649}{106168320}s^2-\frac{74703227}{11796480}st+\frac{124701919}{159252480}t^2\Big] +\frac{1}{\epsilon^2}\Big[\zeta_3\big(-\frac{1371419}{86400}s^2-\frac{236241539}{11059200}st+\frac{4326077}{2764800}t^2\big)+\zeta_2\big(\frac{285899}{124416}s^2+\frac{1058273}{165888}st+\frac{275869}{331776}t^2\big) +\mathrm{S2}\big(\frac{8120143}{663552}s^2+\frac{1893289}{55296}st+\frac{92293}{663552}t^2\big)-\frac{58867708103}{28665446400}s^2+\frac{71191292711}{3185049600}st+\frac{83016363427}{4777574400}t^2\Big] +\frac{1}{\epsilon}\Big[\zeta_5\big(-\frac{1520563}{36864}s^2-\frac{1178767861}{1474560}st-\frac{595491677}{1474560}t^2\big)-\zeta_4\big(\frac{6539029}{921600}s^2+\frac{313837819}{7372800}st+\frac{21665663}{1843200}t^2\big) +\zeta_3\big(\frac{20790944575597}{214990848000}s^2+\frac{6505876281371}{8957952000}st+\frac{70676991239557}{214990848000}t^2\big)+\zeta_2\big(-\frac{491377507}{159252480}s^2-\frac{66476563}{53084160}st +\frac{128393639}{79626240}t^2\big)+\mathrm{T1ep}\big(\frac{8120143}{8957952}s^2+\frac{1893289}{746496}st+\frac{92293}{8957952}t^2\big)+\mathrm{S2}\big(-\frac{14810628499}{159252480}s^2 -\frac{19698937889}{10616832}st-\frac{10272602953}{9953280}t^2\big)+\mathrm{D6}\big(-\frac{616147}{110592}s^2+\frac{1939907}{552960}st+\frac{1299587}{276480}t^2\big) +\frac{9307894793789}{191102976000}s^2+\frac{206124003456599}{573308928000}st+\frac{21562322533673}{143327232000}t^2\Big]$

graphs	(divergence) $\times\, u/(-i/(4\pi)^8\langle 12\rangle^2[34]^2 st A^{\text{tree}}(\frac{\kappa}{2})^{10})$
1–30	$\frac{1}{\epsilon^4}\Big[\frac{1052159}{995328}s^2+\frac{509789}{331776}st-\frac{121001}{497664}t^2\Big]+\frac{1}{\epsilon^3}\Big[\frac{9042569}{1474560}s^2+\frac{34360945}{1327104}st+\frac{73518401}{13271040}t^2\Big] +\frac{1}{\epsilon^2}\Big[\zeta_3\big(-\frac{11443919}{2764800}s^2+\frac{32520079}{552960}st+\frac{5836531}{230400}t^2\big)+\zeta_2\big(\frac{1052159}{497664}s^2+\frac{509789}{165888}st-\frac{121001}{248832}t^2\big) -\mathrm{S2}\big(\frac{637991}{6144}s^2+\frac{10978729}{27648}st+\frac{5080825}{55296}t^2\big)+\big(\frac{270806866183}{7166361600}s^2+\frac{89848068067}{597196800}st+\frac{218093645149}{7166361600}t^2\big)\Big] +\frac{1}{\epsilon}\Big[\zeta_5\big(\frac{100843}{360}s^2+\frac{17118043}{30720}st-\frac{30266471}{92160}t^2\big)+\zeta_4\big(\frac{11435323}{614400}s^2+\frac{232002227}{1843200}st+\frac{22211783}{460800}t^2\big) +\zeta_3\big(\frac{223300432349}{3359232000}s^2-\frac{178732984847}{716636160}st+\frac{951659436383}{53747712000}t^2\big) -\zeta_2\big(\frac{5492357}{245760}s^2+\frac{53468887}{663552}st+\frac{129714599}{6635520}t^2\big)+\mathrm{T1ep}\big(-\frac{637991}{82944}s^2-\frac{10978729}{373248}st-\frac{5080825}{746496}t^2\big) +\mathrm{S2}\big(-\frac{5700088747}{3686400}s^2-\frac{69470348491}{16588800}st-\frac{713512871}{6635520}t^2\big)+\mathrm{D6}\big(-\frac{357421}{43200}s^2-\frac{2891743}{230400}st-\frac{470219}{138240}t^2\big) -\frac{3571506237341}{28665446400}s^2-\frac{1611591325291}{5971968000}st+\frac{2301084608777}{143327232000}t^2\Big]$
31–60	$\frac{1}{\epsilon^4}\Big[-\frac{150715}{82944}s^2-\frac{668333}{221184}st-\frac{7213}{1990656}t^2\Big]+\frac{1}{\epsilon^3}\Big[-\frac{68021833}{13271040}s^2-\frac{36852103}{1327104}st-\frac{298377299}{39813120}t^2\Big] +\frac{1}{\epsilon^2}\Big[\zeta_3\big(-\frac{36448033}{2764800}s^2-\frac{455889533}{2764800}st-\frac{82059281}{1382400}t^2\big)+\zeta_2\big(-\frac{150715}{41472}s^2-\frac{668333}{110592}st-\frac{7213}{995328}t^2\big) +\mathrm{S2}\big(\frac{13910839}{165888}s^2+\frac{1340033}{4096}st+\frac{26303855}{331776}t^2\big)-\frac{68286245653}{2388787200}s^2-\frac{20649690431}{119439360}st-\frac{351701043553}{7166361600}t^2\Big] +\frac{1}{\epsilon}\Big[\zeta_5\big(-\frac{2362679}{9216}s^2-\frac{178668311}{92160}st-\frac{1268313}{10240}t^2\big)+\zeta_4\big(-\frac{124344121}{1843200}s^2-\frac{491722333}{1843200}st-\frac{68141309}{921600}t^2\big) -\zeta_3\big(\frac{630084012997}{53747712000}s^2-\frac{1250670277213}{663552000}st-\frac{6913218302303}{13436928000}t^2\big) +\zeta_2\big(\frac{352368061}{19906560}s^2+\frac{35509679}{663552}st+\frac{227699801}{19906560}t^2\big)+\mathrm{T1ep}\big(\frac{13910839}{2239488}s^2+\frac{1340033}{55296}st+\frac{26303855}{4478976}t^2\big) +\mathrm{S2}\big(\frac{188312318729}{99532800}s^2+\frac{110749829741}{16588800}st+\frac{5056299197}{3981312}t^2\big)+\mathrm{D6}\big(\frac{1220779}{76800}s^2+\frac{44791}{6912}st-\frac{1159831}{230400}t^2\big) +\frac{2755666297013}{28665446400}s^2+\frac{5622513975899}{35831808000}st-\frac{196197363193}{1769472000}t^2\Big]$
61–82	$\frac{1}{\epsilon^4}\Big[\frac{756421}{995328}s^2+\frac{985421}{663552}st+\frac{163739}{663552}t^2\Big]+\frac{1}{\epsilon^3}\Big[-\frac{1670161}{1658880}s^2+\frac{415193}{221184}st+\frac{4863881}{2488320}t^2\Big] +\frac{1}{\epsilon^2}\Big[\zeta_3\big(\frac{110861}{6400}s^2+\frac{16293841}{153600}st+\frac{9408019}{276480}t^2\big)+\zeta_2\big(\frac{756421}{497664}s^2+\frac{985421}{331776}st+\frac{163739}{331776}t^2\big) +\mathrm{S2}\big(\frac{1657459}{82944}s^2+\frac{7734025}{110592}st+\frac{4181095}{331776}t^2\big)-\frac{8243516153}{895795200}s^2+\frac{558349337}{24883200}st+\frac{11133949867}{597196800}t^2\Big] +\frac{1}{\epsilon}\Big[\zeta_5\big(-\frac{1094509}{46080}s^2+\frac{63657091}{46080}st+\frac{5210161}{11520}t^2\big)+\zeta_4\big(\frac{11254769}{230400}s^2+\frac{129860053}{921600}st+\frac{23717743}{921600}t^2\big) -\zeta_3\big(\frac{2745647960587}{53747712000}s^2+\frac{3654260151947}{2239488000}st+\frac{5720906529119}{107495424000}t^2\big) +\zeta_2\big(\frac{11564107}{2488320}s^2+\frac{2244901}{82944}st+\frac{40360999}{4976640}t^2\big)+\mathrm{T1ep}\big(\frac{1657459}{1119744}s^2+\frac{7734025}{1492992}st+\frac{4181095}{4478976}t^2\big) +\mathrm{S2}\big(-\frac{420043}{1215}s^2-\frac{825589625}{331776}st-\frac{5785239343}{4976640}t^2\big)+\mathrm{D6}\big(-\frac{210731}{27648}s^2+\frac{4196129}{691200}st+\frac{1457647}{172800}t^2\big) +\frac{33976742047}{1194393600}s^2+\frac{4046536311847}{35831808000}st+\frac{212357840779}{2239488000}t^2\Big]$

Adds up to zero: no divergence. Enhanced cancellations!
No standard (super)symmetry explanation exists. Origin is a mystery.

$\mathcal{N} = 8$ Supergravity at Five loops

ZB, Carrasco, Chen, Edison, Johansson, Roiban, Parra-Martinez, Zeng (2018)

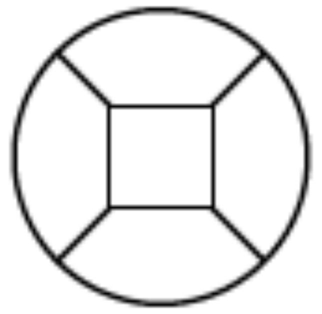


This is UV finite in $D = 4$

Evaluated leading ultraviolet divergence ($D = 24/5$):

diverges don't cancel in $D = 24/5$

$$\mathcal{M}_4^{(5)}\Big|_{\text{leading}} = -\frac{16 \times 629}{25}\left(\frac{\kappa}{2}\right)^{12}(s^2+t^2+u^2)^2\, stu M_4^{\text{tree}} \left(\frac{1}{48}\right.$$



$$+ \frac{1}{16}$$

$$\left.\right)$$

positive definite divergence

Key point: "Impossible" calculations are doable.

Want to harness this for gravitational waves

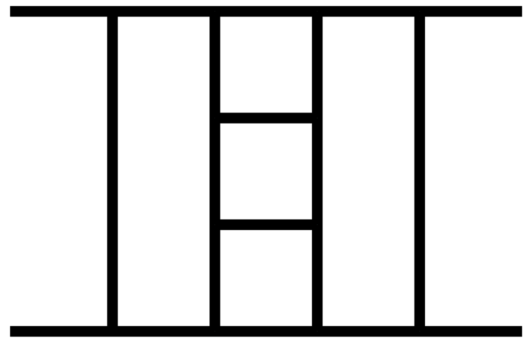
SIDE COMENT: At the time this calculation was rather difficult because no BCJ representation has been found for N=4 SYM 5 loops

$N = 5$ Supergravity at Five Loops?

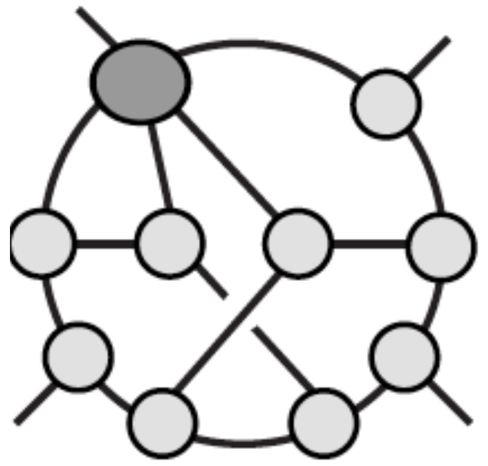
Do the enhanced cancellations continue to 5 loops? Is four loop result an accident?

$N = 5$ sugra: ($N = 4$ sYM) x ($N = 1$ sYM)

5-loop 4-pt $N = 4$ sYM amplitude:



Despite considerable effort, no one has succeeded in finding a BCJ form for the 5 loop $N = 4$ sYM integrand.



double copy easy

BCJ and double copy easy on cuts.
How to combine the cuts for complicated expressions?

New Solution: Identify a basis of (nonplanar) integrand terms.
Basis designed to make "cut term = integrand term".

ZB, Hermann, Roiban, Ruf, Zeng (to appear)

Stay tuned. We are back to working on $N = 5$ sugra at 5 loops. First $N = 1$ sugra at 3 loops.

ZB, Chen, Edison, Gopalka, Hermann, Roiban, Ruf (ongoing)

Summary

Double copy gives us gravity amplitudes from corresponding gauge theory ones.

1. **Useful for high-loop studies of ultraviolet properties of supergravity.**
2. **Next lecture will present applications to gravitational-wave physics.**