Manifesting colour-kinematics duality and the double copy
Homotopy algebras, pure spinors and twistor space

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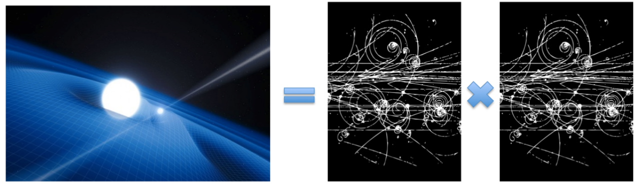
Based on work with: Alexandros Anastasiou, Michael J. Duff, Mia Hughes, Branislav Jurco, Hyungrok Kim, Alessio Marrani, Silvia Nagy, Tommaso Macrelli, Christian Saemann, Martin Wolf and Michele Zoccali

Introduction: gravity as the square of gauge theory



$$g_{\mu\nu} \qquad\qquad A_\mu{}^{a} \qquad\qquad A_\nu{}^{b}$$

- Is gravity the double copy of the other fundamental forces of Nature?
  [Feynman; Papini; Kawai, Lewellen, Tye; Berends, Giele, Kuijf; Bern, Dixon, Dunbar, Perelstein , Rozowsky. . .]

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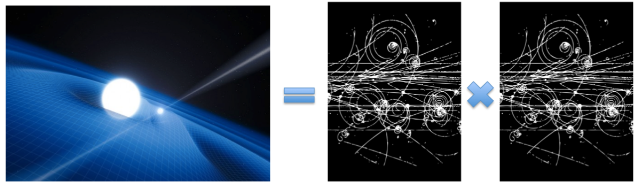

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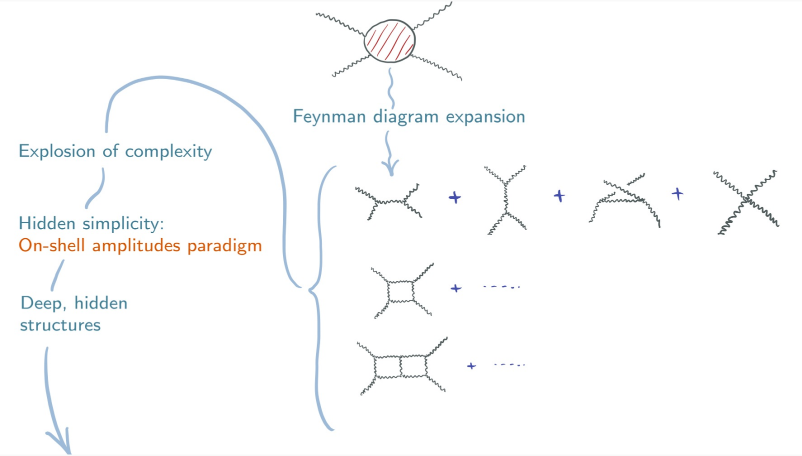
- Renaissance: Bern–Carrasco–Johansson Colour–Kinematics (CK) duality conjecture and double copy of gauge theory and gravity scattering amplitudes
  [Bern, Carrasco, Johansson '08, '10; Bern, Dennen, Huang, Kiermaier '10]

Introduction: colour-kinematics duality

→ Physical observables tested at particle accelerators (e.g. Large Hadron Collider)



→ New insights into the underlying theories themselves

Introduction: colour-kinematics duality

Colour numerators $c_i \sim f_{ab}{}^c f_{cd}{}^e$

$$A_{\mathsf{YM}}^{n,L} = \sum_{i \in \text{cubic diag}} \frac{1}{S_i} \int_L \frac{c_i n_i}{d_i}$$

Kinematic numerators $n_i \sim \varepsilon_\mu p^\mu + \cdots$

Bern-Carrasco-Johansson colour-kinematics duality conjecture 2008:

$$c_i + c_j + c_k = 0 \quad \Rightarrow \quad n_i + n_j + n_k = 0$$

Proven at tree level [Stieberger '09; Bjerrum, Bohr, Damgaard, Vanhove '09; Du, Teng '16; Bridges, Mafra '19; Mizera '19; Reiterer '19. . .]

Conjectured at loop level with highly non-trivial examples [Bern, Carrasco, Johansson '08 '10; Carrasco, Johansson '11; Bern, Davies, Dennen, Huang, Nohle '13; Bern, Davies, Dennen '14. . .]

Assuming colour-kinematics duality is realised, gravity comes for free:

$$A_{\text{glouns}} \sim \sum \int \frac{c_i n_i}{d_i}$$

$$c_i \longrightarrow n_i$$

double copy kinematics

$$A_{\text{gravitons}} \sim \sum \int \frac{n_i n_i}{d_i}$$

[Bern, Carrasco, Johansson '08, '10; Bern, Dennen, Huang, Kiermaier '10]

'Gluons for (almost) nothing, gravitons for free' JJ Carrasco

Introduction: a question

Colour-kinematics duality exposed via powerful on-shell lens cf. Zvi Bern's lectures

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Having discovered colour-kinematics duality on-shell can we now go back off-shell and perhaps learn something new?

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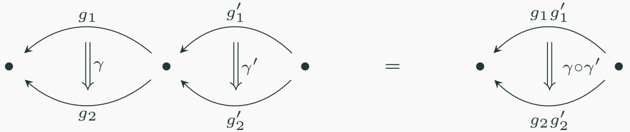
Very hard to see at the level of field theory and action pricinples

Having discovered colour-kinematics duality on-shell can we now go back off-shell and perhaps learn something new?

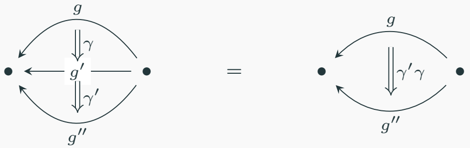
There is a mathematically precise understanding of colour-kinematics duality at the level of actions that can be used to understand new and old examples

Introduction: categorification

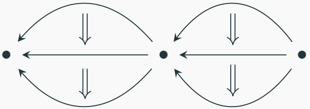
2-arrows form a group under horizontal composition



2-arrows form a groupoid under vertical composition



Interchange law: horizontal and vertical composition are coherent



Lie 2-group → Lie 2-algebra → strict 2-term L_∞-algebra

Introduction: punchline

Colour-kinematics duality

Symmetry (possibly anomalous) of action with kinematic (homotopy) Lie algebra derived from underlying (homotopy) $\mathrm{BV}^{\blacksquare}$-algebra

[Borsten, Jurčo, Kim, Macrelli, Saemann, Wolf (BJKMSW) '20, '21, '22, '23]

- Self-dual (super) Yang–Mills theories in $D = 4$
- (Super) Yang–Mills theories in all dimensions
- M2-brane world–volume theories

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Double copy

Gravity = gauge $\times$ gauge $\to$ tensor product of $\mathrm{BV}^{\blacksquare}$-algebras

[BJKMSW '20, '23; see also Bonezzi, Chiaffrino, Díaz–Jaramillo, Hohm '23]

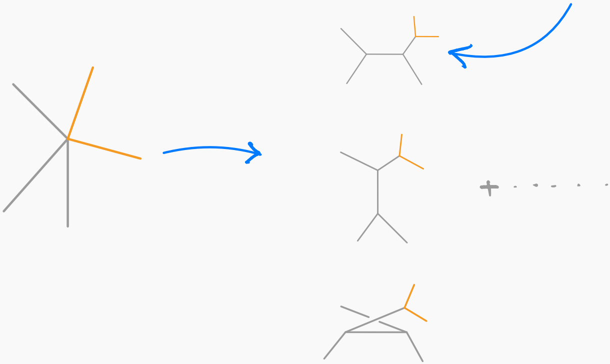
- Bi-form gravity in $D = 2 + 1$
- Cubic pure spinor action for supergravity

Manifesting colour-kinematics duality in the Batalin–Vilkovisky formalism

Manifest colour-kinematics duality of tree-level physical S-matrix

There is a Yang–Mills action such that the Feynman diagrams yield amplitudes manifesting colour-kinematics duality for tree-level amplitudes:

$$A\Box A + \partial AAA + \frac{\Box}{\Box}AAAA + \frac{\partial^3}{\Box^2}AAAAA + \cdots$$



[Bern, Dennen, Huang, Kiermaier '10; Tolotti, Weinzierl '13]

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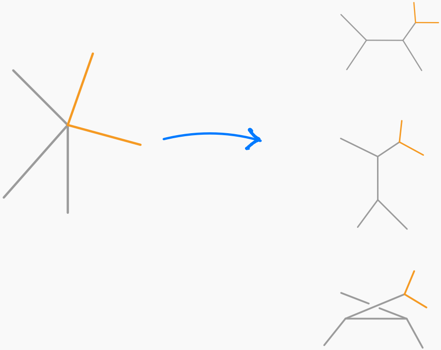
Identically zero by colour Jacobi



[Bern, Dennen, Huang, Kiermaier '10; Tolotti, Weinzierl '13]

Manifest colour-kinematics duality of tree-level physical S-matrix

This can be strictified to have only cubic interactions through infinite tower of auxiliaries [Bern, Dennen, Huang, Kiermaier '10; Tolotti, Weinzierl '13; BJKMSW '21]

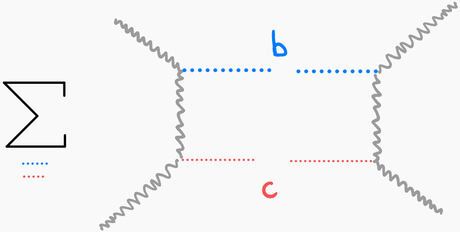
$$\begin{aligned} S^{\mathsf{YM}}_{\text{on-shell CK}} = \operatorname{tr}\int d^Dx\, & \tfrac{1}{2}A_\mu \Box A^\mu + \tfrac{1}{2}g\partial_\mu A_\nu[A^\mu, A^\nu] \\ & + \tfrac{1}{2}B^{\mu\nu\kappa}\Box B_{\mu\nu\kappa} - g(\partial_\mu A_\nu + \tfrac{1}{\sqrt{2}}\partial^\kappa B_{\kappa\mu\nu})[A^\mu, A^\nu] \\ & + C^{\mu\nu}\Box\bar{C}_{\mu\nu} + C^{\mu\nu\kappa}\Box\bar{C}_{\mu\nu\kappa} + C^{\mu\nu\kappa\lambda}\Box\bar{C}_{\mu\nu\kappa\lambda} + \\ & + gC^{\mu\nu}[A_\mu, A_\nu] + g\partial_\mu C^{\mu\nu\kappa}[A_\nu, A_\kappa] - \tfrac{g}{2}\partial_\mu C^{\mu\nu\kappa\lambda}[\partial_{[\nu}A_{\kappa]}, A_\lambda] \\ & + g\bar{C}^{\mu\nu}\big(\tfrac{1}{2}[\partial^\kappa\bar{C}_{\kappa\lambda\mu}, \partial^\lambda A_\nu] + [\partial^\kappa\bar{C}_{\kappa\lambda\nu\mu}, A^\lambda]\big) + \cdots \end{aligned}$$

Purely cubic colour-kinematics duality manifesting Feynman diagrams:

$$A^{n,0}_{\mathsf{YM}} = \sum_i \frac{c_i n_i}{d_i} \quad \text{s.t.} \quad c_i + c_j + c_k = 0 \Rightarrow n_i + n_j + n_k = 0$$

Manifest colour-kinematics duality of tree-level BRST extended S-matrix

To lift to loop-level we should include off-shell unphysical/ghost modes in the external states so that we can glue trees into loops



Extend CK-duality to off-shell unphysical/ghost modes in the external states, the full BRST-extended state space

$$({A_\mu}^a, b^a, c^a, \bar{c}^a)$$

[Anastasiou, LB, Duff, Hughes, Nagy, Zoccali '14 '18; LB, Nagy '20; BJKMSW '20, '21, '22]

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Longitudinal gluons $p_i \cdot \varepsilon_i \neq 0$ on external states $\Rightarrow$

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Compensate for these failures with new BRST-exact vertices [BJKMSW '20]:

$$S^{\mathsf{YM}}_{\text{on-shell BRST-extended CK}} = S^{\mathsf{YM}}_{\text{on-shell CK}} + Q\Psi_{\mathsf{CK}}$$

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$$\begin{aligned}
S^{\text{YM}}_{\text{BRST-extended CK}} &= S^{\text{YM}}_{\text{on-shell CK}} + \int d^D x \tfrac{1}{2} b_a \Box b^a - \bar{c}_a \Box c^a \\
&- K^{\mu}_{1a} \Box \bar{K}^{1a}_{\mu} - K^{\mu}_{2a} \Box \bar{K}^{2a}_{\mu} - g f_{abc} \bar{c}^a \partial^\mu (A^b_\mu c^c) \\
&- \tfrac{1}{2} B^{\mu\nu\kappa}_a \Box B^a_{\mu\nu\kappa} + g f_{abc} \Big(\partial_\mu A^a_\nu + \tfrac{1}{\sqrt{2}} \partial^\kappa B^a_{\kappa\mu\nu} \Big) A^{\mu b} A^{\nu c} \\
&- g f_{abc} \Big\{ K^{a\mu}_1 (\partial^\nu A^b_\mu) A^c_\nu + [(\partial^\kappa A^a_\kappa) A^{b\mu} + \bar{c}^a \partial^\mu c^b] \bar{K}^{1c}_\mu \Big\} \\
&+ g f_{abc} \Big\{ K^{a\mu}_2 \Big[(\partial^\nu \partial_\mu c^b) A^c_\nu + (\partial^\nu A^b_\mu) \partial_\nu c^c \Big] + \bar{c}^a A^{b\mu} \bar{K}^{2c}_\mu \Big\} + \cdots
\end{aligned}$$

Proof is inductive and constructive: BRST colour-kinematics duality up to n-points implies colour-kinematics duality up to $(n+1)$-points

Off-shell colour-kinematics duality

Colour-kinematics duality can be realised as a potentially anomalous symmetry of BV/BRST action [BJKMSW '20, '21, '22]

Tower of auxiliary fields $A^{ia} = (A^{\mu a}, B^{\mu\nu a}, C^{\mu\nu\rho a}, \cdots)$

$$S^{\text{YM}}_{\text{Off-shell CK dual}} = \int C_{ij} c_{ab} A^{ia} \Box A^{jb} + F_{ijk} f_{abc} A^{ia} A^{jb} A^{kc}$$

$$c_{ab} = c_{(ab)} \qquad f_{abc} = f_{[abc]} \qquad c_{a(b} f^{a}_{c)d} = 0 \qquad f_{[ab|d} f^{d}_{c]e} = 0$$

$$C_{ij} = C_{(ij)} \qquad F_{ijk} = F_{[ijk]} \qquad C_{i(j} F^{i}_{k)l} = 0 \qquad F_{[ij|l} F^{l}_{|k]m} = 0$$

Kinematic structure constants mirror colour structure constants

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- ...going off-shell may induce Jacobians → unitarity counterterms

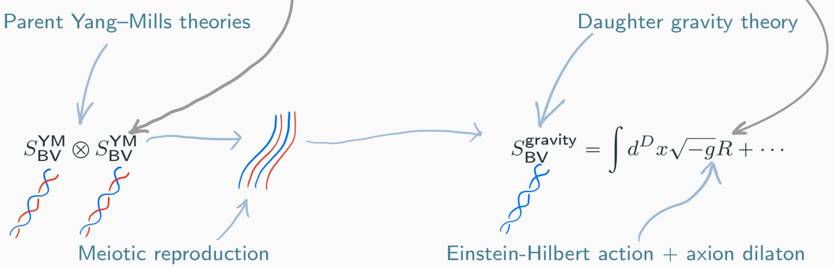
$$\det\left(\mathbb{1} + \frac{\delta f(\phi)}{\delta \phi}\right) = \int \mathcal{D}\bar{\chi}\, \mathcal{D}\chi\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} \int \left(\bar{\chi}_I \chi^I + \bar{\chi}_I \frac{\delta f^I}{\delta \phi^J} \chi^J\right)}$$

⇒ colour-kinematics duality anomaly

Double copy

BRST/BV action double copy **[LB, Nagy '20; BJKMSW '20; '21, '22, '23]**

$$C_{ij}c_{ab}A^{ia}\Box A^{ja}+F_{ijk}f_{abc}A^{ia}A^{jb}A^{kc}\rightarrow C_{ij}\tilde{C}_{\tilde{i}\tilde{j}}A^{i\tilde{i}}\Box A^{j\tilde{j}}+F_{ijk}\tilde{F}_{\tilde{i}\tilde{j}\tilde{k}}A^{i\tilde{i}}A^{j\tilde{j}}A^{k\tilde{k}}$$



Double copy origin of symmetries:

$$\underbrace{(\text{gauge, global susy, R-sym}\ldots)}_{\text{(super) Yang–Mills symmetries}}\longrightarrow\underbrace{(\text{diffeomorphism, local susy, U-duality}\ldots)}_{\text{(super)gravity symmetries}}$$

[Anastasiou, Duff, LB, Hughes, Nagy '14; BJKMSW '20]

Questions

Okay, but. . .

- Proof is constructive and inductive: no theoretical understanding/control over higher vertices or the set of auxiliary fields
- No closed form of colour-kinematics duality manifesting action
- No clue (generically) about the kinematic Lie algebra
- May need non-local field redefinitions $\Rightarrow$ colour-kinematics duality anomaly

So we'd like. . .

- a clear mathematical characterisation of higher vertices
- a closed form colour-kinematics duality manifesting action
- to avoid the need for non-local field redefinitions $\Rightarrow$ perfect all-loop colour-kinematics duality
- an understanding of the kinematic Lie algebra
- a tensor product of some structure that generates double copy

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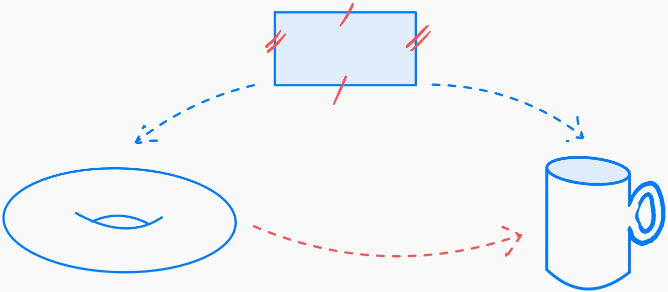
Homotopy algebras and scattering amplitudes

Homotopy Lie algebras

Lie algebra $\mathfrak{g} = (V_0, [-,-])$	L_∞-algebra $\mathfrak{L} = (V, \mu_k)$
Vector space V_0	Graded vector space $\bigoplus_n V_n$
Bracket $\mu_2 = [-,-]$	Higher brackets $\mu_1 = [-],\ \mu_2 = [-,-],\ \mu_3 = [-,-,-], \dots$
Relations Antisymmetry + Jacobi	Homotopy relations Graded antisymmetry + homotopy Jacobi



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Example: A semistrict Lie 2-algebra is a 2-term L_∞-algebra

$$\mathfrak{L} = (V_{-1} \oplus V_0, \mu_k)$$

Differential $\mu_1 = [-]$; Lie bracket $\mu_2 = [-,-]$; Jacobiator $\mu_3 = [-,-,-]$.

$$[[x,y],z] + (-1)^{x(y+z)}[[y,z],x] + (-1)^{y(x+z)}[[x,z],y] = -[[x,y,z]]$$

Homotopy Lie algebras and quantum field theory

Cyclic L_∞-algebra

Every Lagrangian quantum field theory is an L_∞-algebra $\mathfrak{L}_{\text{theory}}$

$$S^{\text{theory}}_{\text{BV}}[\phi,\phi^+] = \sum_k \frac{1}{k!}\langle \phi, \mu_k(\phi,\cdots,\phi)\rangle = \frac{1}{2}\langle \phi, \mu_1(\phi)\rangle + \frac{1}{6}\langle \phi, \mu_2(\phi,\phi)\rangle + \cdots$$

Kinetic term

3-point interaction

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Minimal model theorem

Every $\mathfrak{L}$ is quasi-isomorphic (physically equivalent) to an $\tilde{\mathfrak{L}} \cong (H^\bullet_{\mu_1}(V), \tilde{\mu}_i)$

Cohomology of kinetic operator = space of on shell states

Homotopy Lie algebras and quantum field theory

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Factorisation into homotopy commutative C_∞-algebra

When there is flavour/colour $\mathfrak{g}$, $\mathfrak{L}$ factorises (colour-ordering)

$$\mathfrak{L}_{\text{YM}} = \mathfrak{g} \otimes \mathfrak{C}_{\text{YM}}$$

→ Yang–Mills C_∞-algebra $\mathfrak{C}_{\text{YM}}$ with higher products $m_k(-, -, \cdots -)$

Colour-stripped interactions

Colour-kinematics duality and $\mathrm{BV}^{\blacksquare}_{\infty}$-algebras

Reiterer '18

Colour-kinematics duality of physical tree-level S-matrix is equivalent to a $\mathrm{BV}^{\blacksquare}_{\infty}$-algebra (deformation of BV_{∞}-algebras of [Galvez–Carrillo, Tonks, Vallette '09])

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Off-shell colour-kinematics duality and homotopy algebras

Theory with kinematic Lie algebra $\Leftrightarrow$ a $\mathrm{BV}^{\blacksquare}_{\infty}$-algebra [BJKMSW '21, '22]

$$\mathfrak{C} = \mathfrak{B}$$

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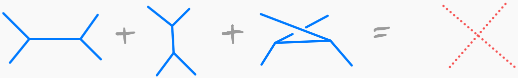
Colour-kinematics duality of physical tree-level S-matrix is equivalent to a $\mathrm{BV}^{\blacksquare}_{\infty}$-algebra (deformation of BV_{∞}-algebras of [Galvez–Carrillo, Tonks, Vallette '09])

Off-shell colour-kinematics duality and homotopy algebras

Theory with kinematic Lie algebra $\Leftrightarrow$ a $\mathrm{BV}^{\blacksquare}_{\infty}$-algebra [BJKMSW '21, '22]

$$\mathfrak{C} = \mathfrak{B}$$

$$[x,[y,z]] + [x,[y,z]] + [x,[y,z]] = 0 \quad \text{up to homotopies}$$



"homotopy Jacobi relations $\Leftrightarrow$ colour-kinematics duality "

See also [Bonezzi, Chiaffrino, Díaz–Jaramillo, Hohm '23] up to four-points

Perfect colour-kinematics duality

Strictification theorem

Every C_∞-algebra is quasi-isomorphic to a commutative algebra with only

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Off-shell colour-kinematics duality implies strict BV$^{\blacksquare}$-algebra

A strict BV$^{\blacksquare}$-algebra $\mathfrak{B}$ is dgca (V, d, m_2) with $\mathsf{b} : V \to V$ such that

$$\mathsf{b}^2 = 0, \qquad \blacksquare \;:=\; \mathsf{d} \circ \mathsf{b} + \mathsf{b} \circ \mathsf{d}$$

and b is second order w.r.t $\mathsf{m}(-,-)$ so that

$$[x, y] = \mathsf{b} m_2(x, y) - m_2(\mathsf{b}x, y) - (-1)^x m_2(x, \mathsf{b}y)$$

is a (shifted) Lie bracket: the kinematic Lie algebra

The Chern-Simons paradigm

Chern–Simons theory has off–shell CK duality ⇒ Chern–Simons has a BV$^\square$-algebra

[Ben–Shahar, Johansson '21; BJKMSW '22]

$$S^{\mathsf{CS}}_{\mathsf{BV}} = \int \mathrm{tr}\Big(\tfrac{1}{2} A \wedge \mathrm{d}A + \tfrac{1}{3!} A \wedge [A,A] + A^+ \wedge (\mathrm{d}c + [A,c]) + \tfrac{1}{2} c^+ \wedge [c,c]\Big)$$

$$\mathfrak{B}_{\mathsf{CS}} = \Omega^0 \underset{\mathrm{d}^\dagger}{\overset{\mathrm{d}}{\rightleftarrows}} \Omega^1 \underset{\mathrm{d}^\dagger}{\overset{\mathrm{d}}{\rightleftarrows}} \Omega^2 \underset{\mathrm{d}^\dagger}{\overset{\mathrm{d}}{\rightleftarrows}} \Omega^3$$

$$\mathsf{d}A = \mathrm{d}A, \quad \mathsf{b}A = \mathrm{d}^\dagger A, \quad m_2(A,B) = A \wedge B$$

$$\mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger \mathrm{d} = \square$$

The Chern-Simons paradigm

Kinematic Lie algebra given by derived bracket

$$(-1)^p[\alpha, \beta] = -\mathrm{d}^\dagger(\alpha \wedge \beta) + \mathrm{d}^\dagger\alpha \wedge \beta + (-1)^p \alpha \wedge \mathrm{d}^\dagger\beta$$

is Schouten–Nijenhuis algebra of totally antisymmetric tensor fields, the natural Gerstenhaber algebra on three-dimensional Minkowski space [BJKMSW '22]

Restricting to fields yields diffeomorphism algebra identified in [Ben–Shahar, Johansson '21]

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Look for Chern-Simons-type actions!

Holomorphic Chern-Simons theory on twistor space

Self-dual super Yang–Mills theory is equivalent to holomorphic Chern–Simons theory on super twistor space $Z \cong \mathbb{R}^{4|8} \times \mathbb{C}P^1$ with local coordinates $(x^\mu, \eta^i, \lambda^\alpha)$

$$S_{\mathsf{hCS}} = \int \Omega \wedge \mathrm{tr}\Big(\tfrac{1}{2} A \wedge \bar{\partial}_{\mathrm{red}} A + \tfrac{1}{3!} A \wedge [A, A] + A^+ \wedge (\bar{\partial}_{\mathrm{red}} c + [A, c]) + \tfrac{1}{2} c^+ \wedge [c, c]\Big),$$

$$\bar{\partial}_{\mathrm{red}} = \hat{e}^\alpha \hat{E}_\alpha + \hat{e}^0 \hat{E}_0, \qquad \mathsf{b} = -\frac{4}{|\lambda|^2} \varepsilon^{\alpha\beta} \iota_{E_\alpha} \iota_{\hat{E}_\beta} \partial_{\mathrm{red}} + 2\varepsilon^{\alpha\beta} \iota_{\hat{E}_\alpha} \iota_{\hat{E}_\beta} \hat{e}^0 \wedge$$

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- Kaluza–Klein expansion on $\mathbb{C}P^1$ gives infinite tower of auxiliary fields required for colour-kinematics duality

$$A^a(x, \eta, \lambda) \sim A(x, \eta)^a + A(x, \eta)^{\alpha a} \lambda_\alpha + A(x, \eta)^{\alpha\beta a} \lambda_\alpha \lambda_\beta + \cdots$$

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- Reproduces the kinematic Lie algebra of area-preserving diffeomorphisms on $\mathbb{C}^2$ identified in [Monteiro, O'Connell '11] cf. [Bonezzi, Diaz–Jaramillo, Nagy '23]

- Generalise to full Yang-Mills using ambitwistor space (to appear [BJKMSW '24])

Pure spinors and tree-level colour-kinematics duality

Pure spinor actions:

$$\int \Omega \mathrm{tr}\Big(\Psi Q \Psi + \tfrac{1}{3}\Psi\Psi\Psi\Big)$$

Pure spinors and tree-level colour-kinematics duality

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- Super Yang–Mills tree-level colour–kinematics duality (b-ghost divergences obstruct loop-level proof) [Ben–Shahar, Guillum '21; BJKMSW '23]
- Bagger–Lambert–Gustavsson and ABJM theories of M2-branes have tree-level colour-kinematics duality [BJKMSW '23] as conjectured in [Bargheer, He, McLoughlin '12; Huang, Johansson '12]
- Double coy: cubic pure spinor actions for supergravity [BJKMSW '23]
- Natural conjecture: colour-kinematics duality for open string field theory

Future directions

Curved backgrounds and classical double copy (beyond perturbation theory?)

Cf. **[Monteiro, O'Connell, White '14; Cardoso, Nagy, Nampuri '16; Luna, Monteiro, Nicholson, Ochirov, O'Connell, Westerberg, White '16; Berman, Chacón, Luna, White '18; Kosower, Maybee, O'Connell '18; Bern, Cheung, Roiban, Shen, Solon, Zeng '19; Bern, Luna, Roiban, Shen, Zeng '20; Chacón-Nagy, White '21; Adamo, Cristofoli, Ilderton '22 Lipstein, Nagy '23...]**

Extra material for questions

Homotopy algebras

Consider a cochain complex $(C^\bullet, \mathsf{d})$

$$\cdots \xrightarrow{\quad \mathsf{d} \quad} C^i \xrightarrow{\quad \mathsf{d} \quad} C^{i+1} \xrightarrow{\quad \mathsf{d} \quad} C^{i+2} \xrightarrow{\quad \mathsf{d} \quad} \cdots$$

$\mathsf{d}^2 = 0$ with some compatible algebraic structure (“mulitplication” map m)

$$\mathsf{m} : C^i \times C^j \to C^{i+j}; \quad (x, y) \mapsto \mathsf{m}(x, y)$$

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Example: Hodge–de Rham complex $\Omega^\bullet(M)$ of i-forms with exterior derivative

$$\mathsf{m}(A_i, A_j) = A_i \wedge A_j = (-)^{ij} A_j \wedge A_i, \quad \mathsf{d}(A_i \wedge A_j) = \mathsf{d}A_i \wedge A_j + (-)^i A_i \wedge \mathsf{d}A_j$$

is a differential graded commutative algebra (dgca)

Homotopy algebras

Given a morphism $\varphi : (C^\bullet, \mathsf{d}) \to (\tilde{C}^\bullet, \tilde{\mathsf{d}})$

$$
\begin{array}{ccccccccc}
\cdots & \xrightarrow{\mathsf{d}} & C^i & \xrightarrow{\mathsf{d}} & C^{i+1} & \xrightarrow{\mathsf{d}} & C^{i+2} & \xrightarrow{\mathsf{d}} & \cdots \\
 & & \downarrow{\scriptstyle \varphi_i} & & \downarrow{\scriptstyle \varphi_{i+1}} & & \downarrow{\scriptstyle \varphi_{i+2}} & & \\
\cdots & \xrightarrow[\tilde{\mathsf{d}}]{} & \tilde{C}^i & \xrightarrow[\tilde{\mathsf{d}}]{} & \tilde{C}^{i+1} & \xrightarrow[\tilde{\mathsf{d}}]{} & \tilde{C}^{i+2} & \xrightarrow[\tilde{\mathsf{d}}]{} & \cdots
\end{array}
$$

Q: Can the algebraic structure m on $(C^\bullet, \mathsf{d})$ also be transferred to an algebraic structure $\tilde{\mathsf{m}}$ on $(\tilde{C}^\bullet, \tilde{\mathsf{d}})$?

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$$

Q: Can the algebraic structure m on $(C^\bullet, \mathsf{d})$ also be transferred to an algebraic structure $\tilde{\mathsf{m}}$ on $(\tilde{C}^\bullet, \tilde{\mathsf{d}})$?

A: Yes, if we allow for a richer homotopy algebraic structure

Homotopy algebras

Algebraic identities (e.g. associativity, commutativity or Jacobi) hold only up to cochain homotopies

→ tower of higher products $\mathsf{d}(x) = \mathsf{m}_1(x), \mathsf{m}_2(x, y), \mathsf{m}_3(x, y, z), \ldots$

$$\mathsf{m}_n : C^{i_1} \times C^{i_2} \times \cdots \times C^{i_n} \to C^{i_1+i_2+\cdots i_n-n+2}$$

Informally: generalise familiar algebras to include higher products satisfying higher relations up to homotopies:

Associative algebras	→	homotopy associative A_∞-algebras [Stasheff '63]
Commutative algebras	→	homotopy commutative C_∞-algebras [Kadeishvili '82]
Lie algebras	→	homotopy Lie L_∞-algebras [Zwiebach '93; Hinich, Schechtman '93]

Homotopy Lie algebras: higher products and relations

ALGEBRA + HOMOTOPY = OPERAD [Valette '12]:

L_∞-algebras are given by degree one differential derivations on $\mathcal{L}ie^!((V[1])^*)$ for some graded vector space V

Operads are the appropriate mathematical arena for constructing homotopy algebras

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Operads are the appropriate mathematical arena for constructing homotopy algebras

Unpacking this definition: an L_∞-algebra $\mathfrak{L}$ is a graded vector space $V \cong \bigoplus_i V_i$ together with graded anti-symmetric i-linear maps

$$\mu_i : V \times \cdots \times V \to V$$

of degree $2 - i$ that satisfy the homotopy Jacobi identities

$$\sum_{\substack{i=j+k \\ \sigma \in \overline{\mathrm{Sh}}(j,k;i)}} (-1)^k \chi_{(\sigma; v_1, \dots, v_i)} \mu_{k+1}\big(\mu_j(v_{\sigma(1)}, \dots, v_{\sigma(j)}), v_{\sigma(j+1)}, \dots, v_{\sigma(i)}\big) = 0$$

Homotopy Lie algebras: higher products and relations

The first three homotopy Jacobi identities are

$$\mu_1(\mu_1(v_1)) \;=\; 0$$

$$\mu_1(\mu_2(v_1,v_2)) \;=\; \mu_2(\mu_1(v_1),v_2) + (-1)^{|v_1|}\mu_2(v_1,\mu_1(v_2))$$

$$\begin{aligned}\mu_2(\mu_2(v_1,v_2),v_3) + (-1)^{|v_1|\,|v_2|}\mu_2(v_2,\mu_2(v_1,v_3)) - \mu_2(v_1,\mu_2(v_2,v_3))& \\ = \; \mu_1(\mu_3(v_1,v_2,v_3)) + \mu_3(\mu_1(v_1),v_2,v_3) + (-1)^{|v_1|}\mu_3(v_1,\mu_1(v_2),v_3)& \\ + (-1)^{|v_1|+|v_2|}\mu_3(v_1,v_2,\mu_1(v_3))&\end{aligned}$$

- The unary product μ_1 is a differential and a derivation with respect to the binary product μ_2
- The ternary product μ_3 captures the failure of the binary product μ_2 to satisfy the standard Jacobi identity