ORTA DOĞU TEKNİK ÜNİVERSİTESİ
MIDDLE EAST TECHNICAL UNIVERSITY

2 NOVEL WAYS TO COMPUTE THE SURFACE GRAVITY : Middle East Technical University, Turkey

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Temperature of A Black Hole

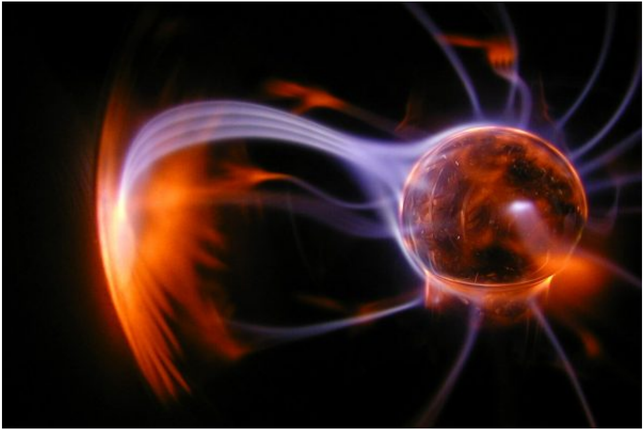

Figure: if we throw the Sun into a black hole what would happen to its temperature?

Outline

Introduction

"The law that entropy always increases holds, I think, the supreme position among the laws of Nature. If someone points out to you that your pet theory of the universe is in disagreement with Maxwell's equations - then so much the worse for Maxwell's equations. If it is found to be contradicted by observation - well, these experimentalists do bungle things sometimes. But if your theory is found to be against the Second Law of Thermodynamics I can give you no hope; there is nothing for it to collapse in deepest humiliation."

Arthur Eddington, New Pathways in Science

Hawking Temperature

$$T_H = \frac{\hbar c^3}{8\pi GMk_B} = \frac{\kappa}{2\pi} \tag{1}$$

$$\zeta^\mu \nabla_\mu \zeta^\nu = -\kappa \zeta^\nu \tag{2}$$

- if the integral curves of the null Killing vector ζ^μ are restricted to be affinely parameterized, then $\nabla_\zeta \zeta = 0$ and κ disappears. So affine parameterization should not be imposed.
- a constant scaling of $\zeta^\mu \to a\zeta^\mu$, also scales $\kappa \to a\kappa$

The $\mathcal{P}$-tensor

$$\begin{aligned}\mathcal{P}^{\nu}{}_{\mu\beta\sigma} :=& R^{\nu}{}_{\mu\beta\sigma} + \delta^{\nu}_{\sigma}\mathcal{G}_{\beta\mu} - \delta^{\nu}_{\beta}\mathcal{G}_{\sigma\mu} + \mathcal{G}^{\nu}_{\sigma}g_{\beta\mu} - \mathcal{G}^{\nu}_{\beta}g_{\sigma\mu} \\ &+ (\frac{R}{2} - \frac{\Lambda(n+1)}{n-1})(\delta^{\nu}_{\sigma}g_{\beta\mu} - \delta^{\nu}_{\beta}g_{\sigma\mu}).\end{aligned} \tag{3}$$

where $\mathcal{G}^{\nu}_{\beta} := R^{\nu}_{\beta} - \frac{1}{2}R\delta^{\beta}_{\nu} + \Lambda\delta^{\beta}_{\nu}$.

- contraction yields the Einstein tensor, $\mathcal{P}^{\nu}{}_{\mu v\sigma} = (3-n)\mathcal{G}_{\mu\sigma}$.
- does not obey $\nabla_{[\mu}\mathcal{P}_{\rho\nu]\beta\sigma} \neq 0$, but for all of its indices

$$\nabla_{\nu}\mathcal{P}^{\nu}{}_{\mu\beta\sigma} = 0. \tag{4}$$

$$\nabla_\nu(\mathcal{P}^{\nu\mu}{}_{\beta\sigma}\mathcal{F}^{\beta\sigma}) = \mathcal{P}^{\nu\mu}{}_{\beta\sigma}\nabla_\nu\mathcal{F}^{\beta\sigma}. \tag{5}$$

its potential as

$$\mathcal{F}^{\beta\sigma} = \frac{1}{2}\left(\nabla^\beta\chi^\sigma - \nabla^\sigma\chi^\beta\right), \tag{6}$$

and decompose χ^σ as follows

$$\chi^\sigma := \xi^\sigma + \psi^\sigma, \tag{7}$$

where ξ^σ is a Killing vector (i.e. $\nabla^\beta\xi^\sigma + \nabla^\sigma\xi^\beta = 0$) and ψ^σ is a generic vector

$$\mathcal{P}^{\nu\mu}{}_{\beta\sigma}R^{\sigma\beta}{}_{\nu}{}^{\lambda} = -\frac{1}{2}\mathcal{H}^{\mu\lambda} - \frac{1}{4}g^{\mu\lambda}\chi_{GB} + \frac{2\Lambda(n-3)}{(n-1)}R^{\mu\lambda}. \qquad (8)$$

valid for any smooth metric

$$\nabla_{\nu}(\mathcal{P}^{\nu\mu}{}_{\beta\sigma}\nabla^{\beta}\xi^{\sigma}) = \Big(\frac{2\Lambda(n-3)}{(n-1)}R^{\mu\lambda} - \frac{1}{2}\mathcal{H}^{\mu\lambda} - \frac{1}{4}g^{\mu\lambda}\chi_{GB}\Big)\xi_{\lambda}. \qquad (9)$$

Construction of Geometrical Identity

$$\nabla_\mu \nabla_\nu (\mathcal{P}^{\nu\mu}{}_{\beta\sigma} \nabla^\beta \xi^\sigma) = \nabla_\mu \Big(\frac{2\Lambda(n-3)}{(n-1)} R^{\mu\lambda} - \frac{1}{2}\mathcal{H}^{\mu\lambda} - \frac{1}{4} g^{\mu\lambda} \chi_{GB} \Big) \xi_\lambda. \tag{10}$$

Let us concentrate on the left-hand side which reads

$$\begin{aligned}
\nabla_\mu \nabla_\nu (\mathcal{P}^{\nu\mu}{}_{\beta\sigma} \nabla^\beta \xi^\sigma) &= \frac{1}{2}[\nabla_\mu, \nabla_\nu](\mathcal{P}^{\nu\mu}{}_{\beta\sigma} \nabla^\beta \xi^\sigma) \\
&= R_{\mu\nu}{}^{\nu}{}_{\lambda}(\mathcal{P}^{\lambda\mu}{}_{\beta\sigma} \nabla^\beta \xi^\sigma) \\
&+ R_{\mu\nu}{}^{\mu}{}_{\lambda}(\mathcal{P}^{\nu\lambda}{}_{\beta\sigma} \nabla^\beta \xi^\sigma) \\
&= -R_{\mu\lambda}(\mathcal{P}^{\lambda\mu}{}_{\beta\sigma} \nabla^\beta \xi^\sigma) \\
&\quad + R_{\nu\lambda}(\mathcal{P}^{\nu\lambda}{}_{\beta\sigma} \nabla^\beta \xi^\sigma).
\end{aligned} \tag{11}$$

$$\nabla_\mu \mathcal{J}^\mu = 0$$

$$\mathcal{J}^\mu := \nabla_\nu (\mathcal{P}^{\nu\mu}{}_{\beta\sigma} \nabla^\beta \xi^\sigma) \tag{12}$$

and

$$\mathcal{J}^\mu = \Big(\frac{2\Lambda(n-3)}{(n-1)} R^{\mu\lambda} - \frac{1}{2}\mathcal{H}^{\mu\lambda} - \frac{1}{4} g^{\mu\lambda}\chi_{GB} \Big) \xi_\lambda . \tag{13}$$

$$\mathcal{J}^{\mu} = \nabla_{\nu}(\mathcal{P}^{\nu\mu}{}_{\beta\sigma}\nabla^{\beta}\xi^{\sigma}) = -\frac{1}{4}\xi^{\mu}R_{\rho\alpha\beta\sigma}R^{\rho\alpha\beta\sigma}. \tag{14}$$

$$\nabla_{\mu}\mathcal{J}^{\mu} = 0 \tag{15}$$

yields a true conservation law

$$\partial_{\mu}(\sqrt{-g}\,\mathcal{J}^{\mu}) = 0 \tag{16}$$

$$\int_{\Sigma} d^3y\sqrt{\gamma}\,n_{\mu}\nabla_{\nu}(R^{\nu\mu}{}_{\beta\sigma}\nabla^{\beta}\xi^{\sigma}) = -\frac{1}{4}\int_{\Sigma} d^3y\sqrt{\gamma}\,n_{\mu}\xi^{\mu}R_{\rho\alpha\beta\sigma}R^{\rho\alpha\beta\sigma}, \tag{17}$$

Surface Gravity

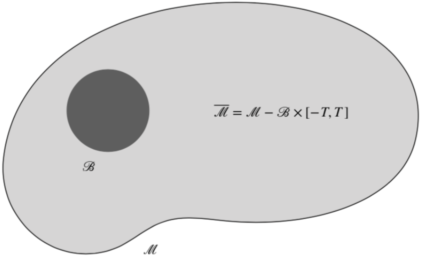

Figure: $\mathcal{M}$ denotes the four (or generically $n > 3$) dimensional spacetime, $\mathcal{B}$ represents the three (or generically $n - 1$) dimensional ball for which the boundary is the cross section of the event horizon. Also, $\bar{\mathcal{M}} = \mathcal{M} - \mathcal{B} \times [-T, T]$ denotes the region of the spacetime between the event horizon and the boundary of the black hole at infinity.

use the Stokes' theorem on the left-hand side as follows

$$\int_{\Sigma} d^3y\sqrt{\gamma}\, n_\mu \nabla_\nu (R^{\nu\mu}{}_{\beta\sigma}\nabla^\beta \xi^\sigma) = \int_{\partial\Sigma} d^2z\sqrt{\gamma^{(\partial\Sigma)}}\, n_\mu \sigma_\nu R^{\nu\mu}{}_{\beta\sigma}\nabla^\beta \xi^\sigma \quad (18)$$

where $\partial\Sigma$ is the (spacelike) boundary of the spacelike surface Σ while σ_ν is its spacelike outward unit normal vector and $\gamma^{(\partial\Sigma)}_{\mu\nu} := g_{\mu\nu} + n_\mu n_\nu - \sigma_\mu \sigma_\nu$ is the induced metric on it.

Introducing the antisymmetric binormal as

$$\epsilon_{\mu\nu} := \frac{1}{2}\left(n_\mu \sigma_\nu - n_\nu \sigma_\mu\right), \tag{19}$$

$$\boxed{\int_{\partial\Sigma} d^2 z \sqrt{\gamma^{(\partial\Sigma)}}\, \epsilon_{\mu\nu} R^{\nu\mu}{}_{\beta\sigma} \nabla^\beta \xi^\sigma = -\frac{1}{4} \int_{\Sigma} d^3 y \sqrt{\gamma}\, n_\mu \xi^\mu R_{\rho\alpha\beta\sigma} R^{\rho\alpha\beta\sigma}.} \tag{20}$$

$$\zeta = \partial_t + \Omega_H \partial_\phi, \tag{21}$$

which is the horizon-generating null Killing vector field. Here Ω_H is the angular velocity of the event horizon given as

$$\Omega_H = \frac{a}{r_H^2 + a^2}, \tag{22}$$

which makes $\zeta^\mu \zeta_\mu = 0$ on the event horizon
So, using (21) in (2) one arrives at the known result for the surface gravity of the Kerr black hole

$$\kappa = \frac{r_H^2 - a^2}{2r_H(r_H^2 + a^2)}, \tag{23}$$

and the Hawking temperature follows from (1)

$$\mathcal{E}\left[\partial_t\right] = -\frac{16\pi r_H m^2 (r_H^2 - a^2)}{(r_H^2 + a^2)^3}. \tag{24}$$

$$\kappa = -\frac{1}{32\pi}\left(\frac{a}{m r_H \Omega_H}\right)^2 \mathcal{E}\left[\partial_t\right], \tag{25}$$

which is equivalent to (23). For the Schwarzschild black hole, $a = 0$ and one finds the correct limit $\kappa = \frac{1}{4m}$.

Conclusion and Further Works

Our formulation is geometric in the sense that it is valid for any gravity theory, for any n 4 dimensions. The contents of a theory enter only after the geometric identity.

New geometric Identity can be used to obtain surface gravity κ for non-stationary spacetimes

The new definition can be used on Vaidya Spacetime

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Thank You

for your attention.

Do you have any question?