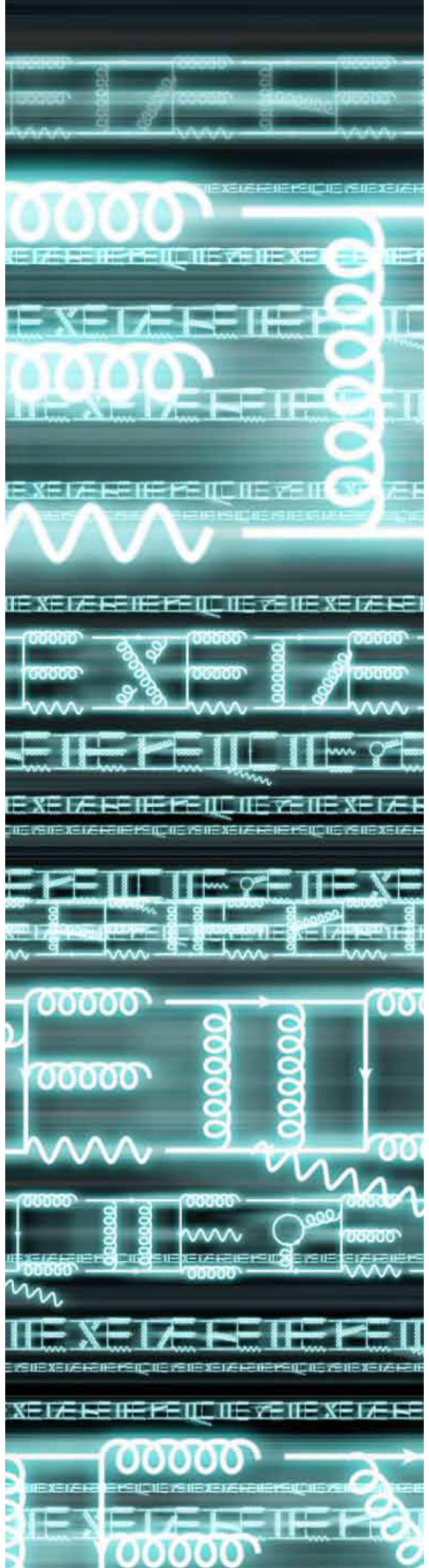


Scattering Amplitude: Theory and Applications Gravitational Waves

Zvi Bern

June 22, 2024

Erice School

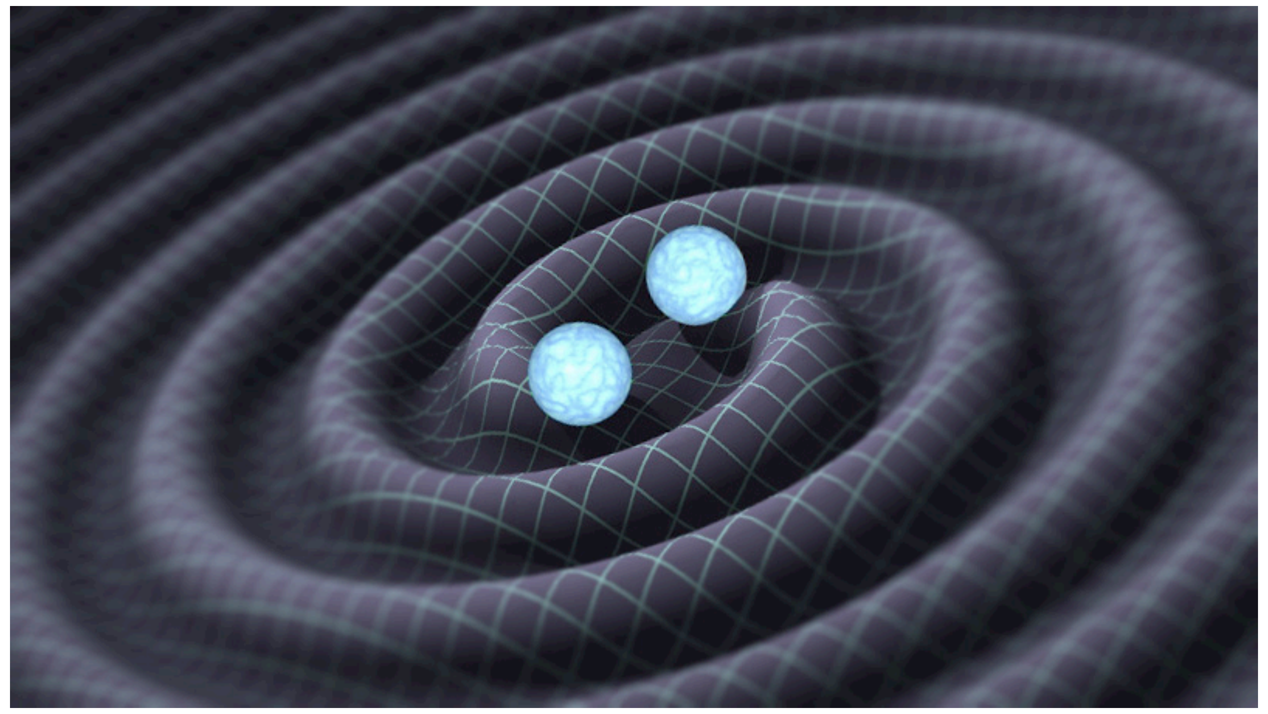
ZB, C. Cheung, R. Roiban, C. H. Shen, M. Solon, M. Zeng, arXiv:1901.04424 and arXiv:1908.01493.
ZB, A. Luna, R. Roiban, C. H. Shen, M. Zeng, arXiv:2005.03071
ZB, J. Parra-Martinez, R. Roiban, M. Ruf. C.-H. Shen, M. Solon, M. Zeng, arXiv:2101.07254; arXiv:2112.10750
ZB, Kosmopoulos, Luna, Roiban, Teng, arXiv: 2203.06202.
ZB, Herrmann, Roiban, Ruf, Smirnov, Smirnov, Zeng: arXiv:2305.08981; arXiv:2406.01554
ZB, Kosmopoulos, Luna, Roiban, Sheopner, Teng, Vines, 2308.14176

UCLA Mani L. Bhaumik Institute for Theoretical Physics

Outline

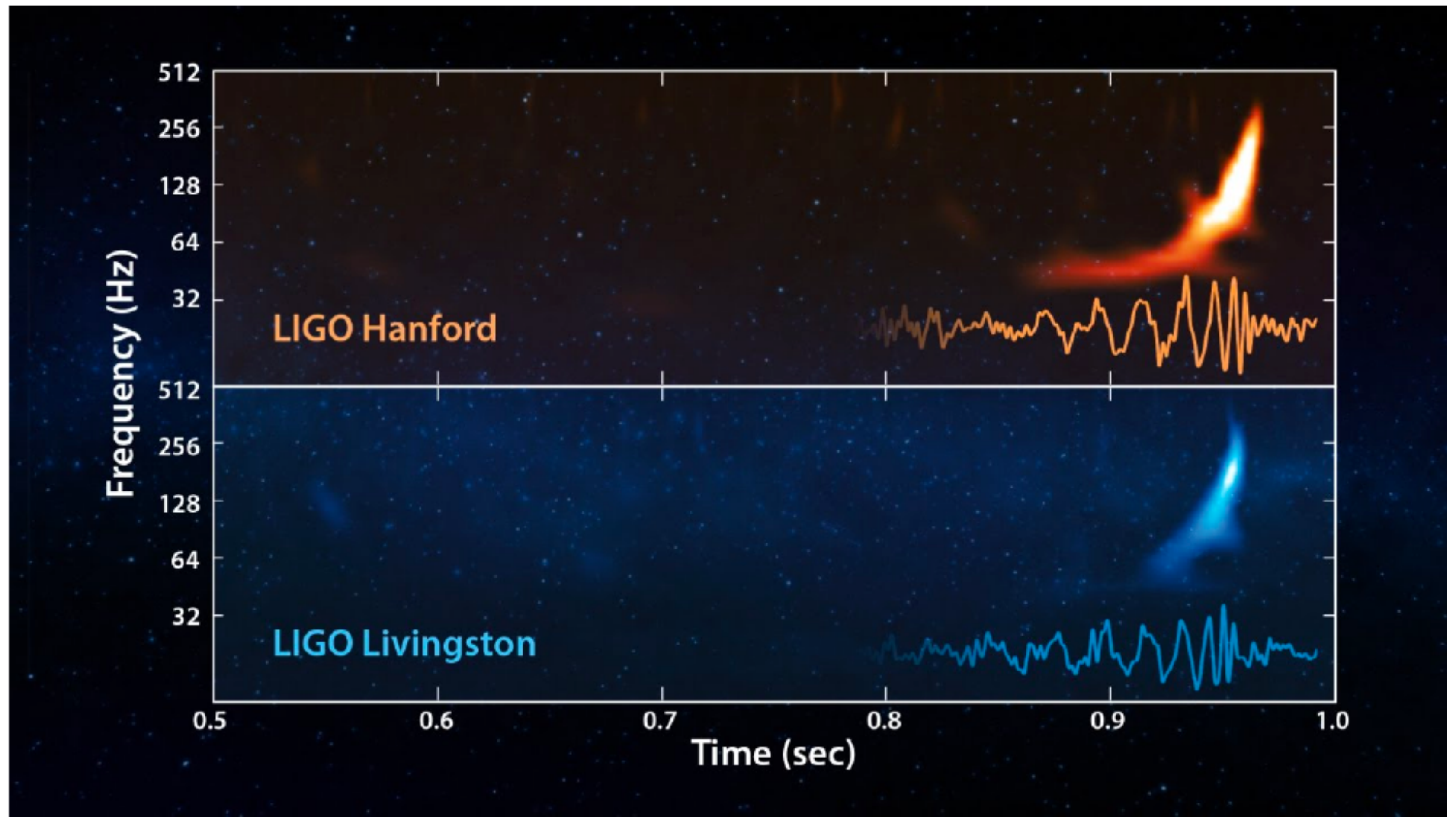
Here we discuss the applications to gravitational-wave physics.

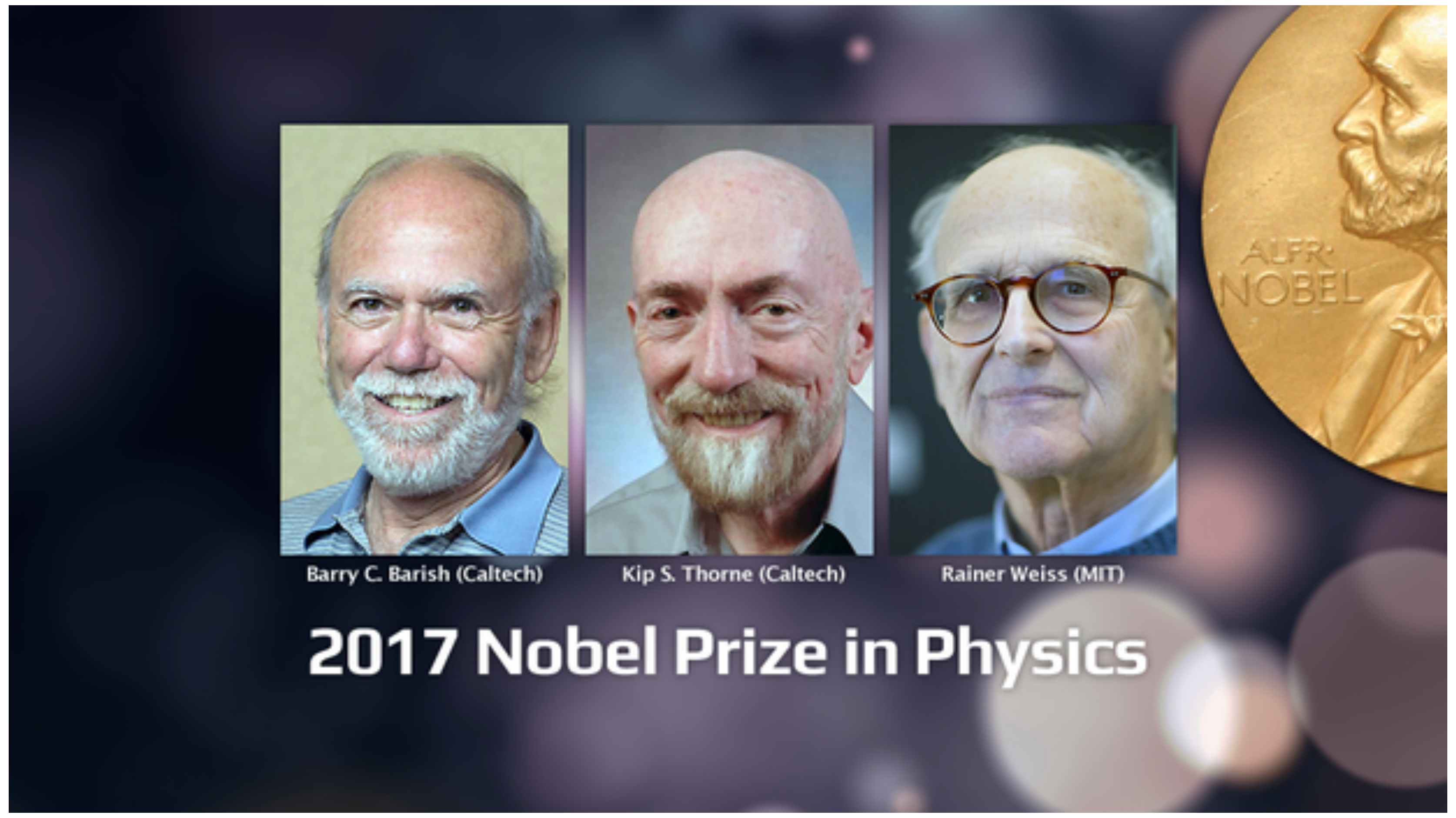
1. **Overview. What do quantum scattering amplitudes have to do with classical gravitational waves.**
2. **Brief review of basics.**
3. **Sample application: High precision gravitational-wave calculations.**
4. **Conclusions and Outlook.**



Outline

Era of gravitational-wave astronomy has begun.



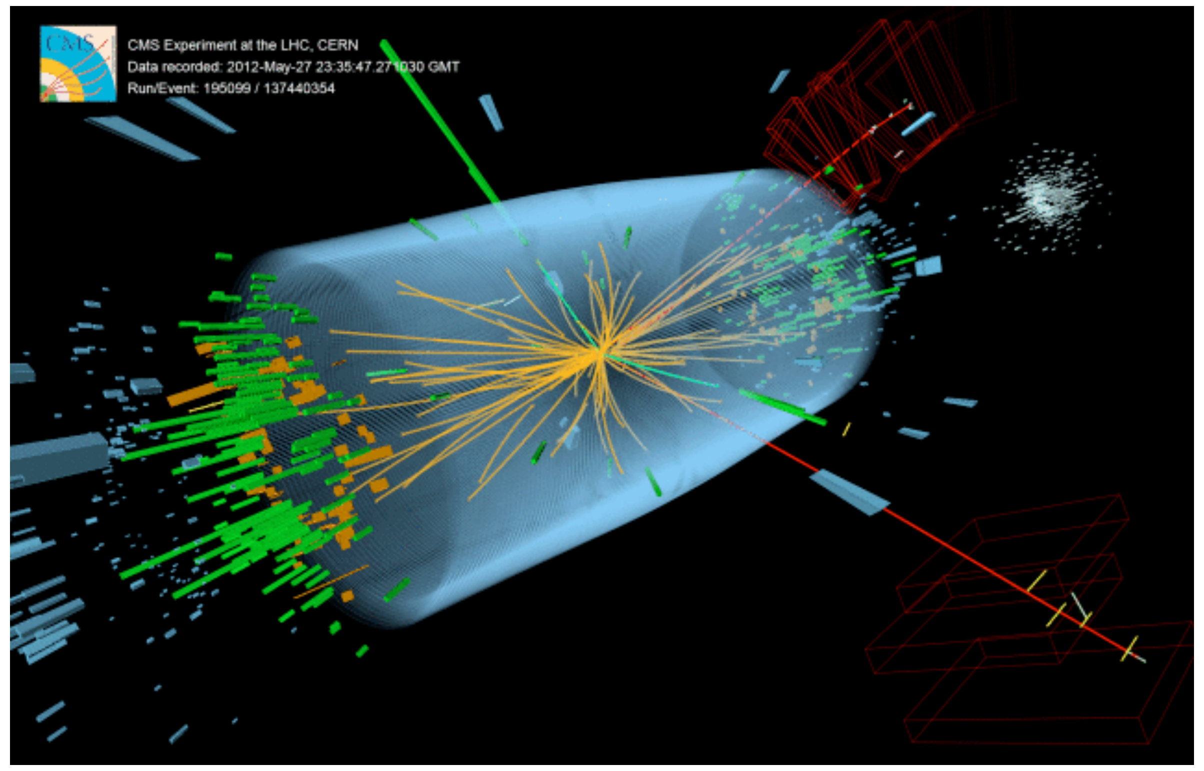


How can we, in the amplitudes community, help out with core mission of gravitational wave detectors ?

Can Scattering Amplitude Help with Gravitational Wave Theory?

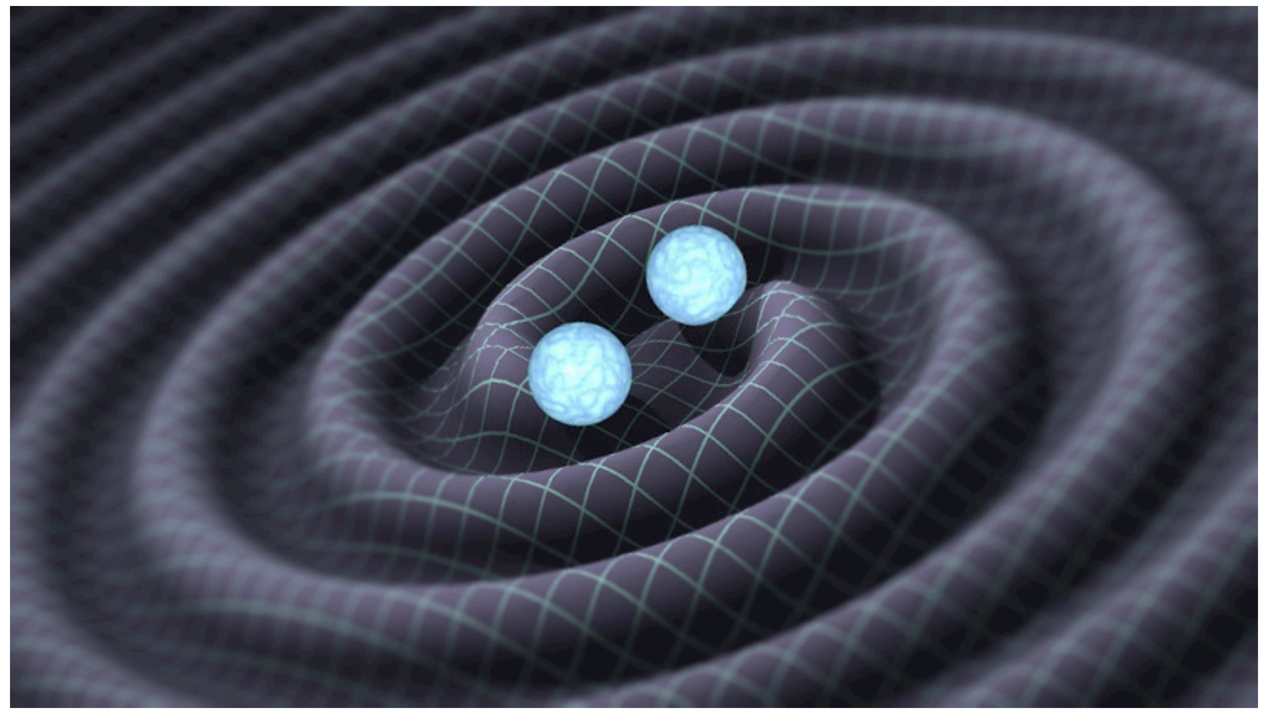
What does particle physics have to do with classical dynamics of astrophysical objects?

unbounded trajectory



gauge theories, QCD, electroweak
quantum field theory

bounded orbit



General Relativity
classical physics

Black holes and neutron stars are point particles as far as long-wavelength radiation is concerned.

Iwasaki (1971); **Goldberger, Rothstein (2006)**; Vaydia, Foffa , Porto, Rothstein, Sturani;
Kol; Bjerrum-Bohr, Donoghue, Holstein, Plante, Pierre Vanhove; Levi, Steinhoff; Vines etc

Will explain that scattering amplitudes are well suited for perturbative gravitational wave calculations in post-Minkowskian framework.

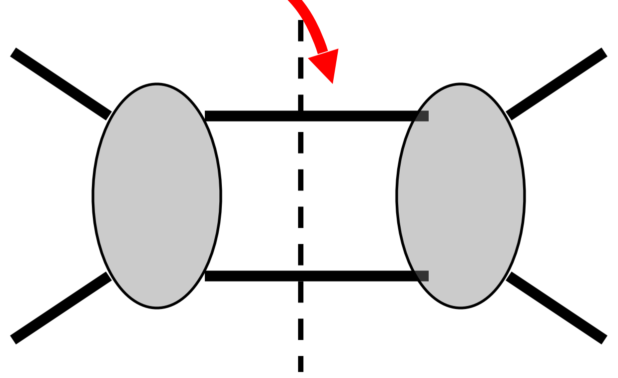
From Last Lecture: Generalized Unitarity Method

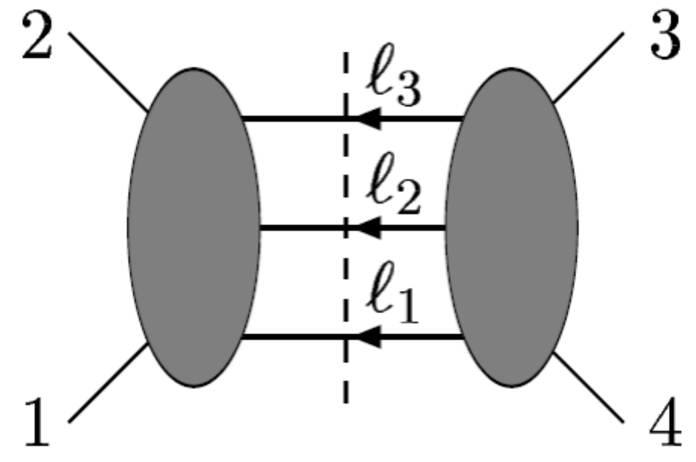
Use tree amplitudes to build higher order (loop) amplitudes.

$$E^2 = \vec{p}^{\,2} + m^2$$ ← **on-shell**

ZB, Dixon, Dunbar and Kosower (1994)

Two-particle cut:



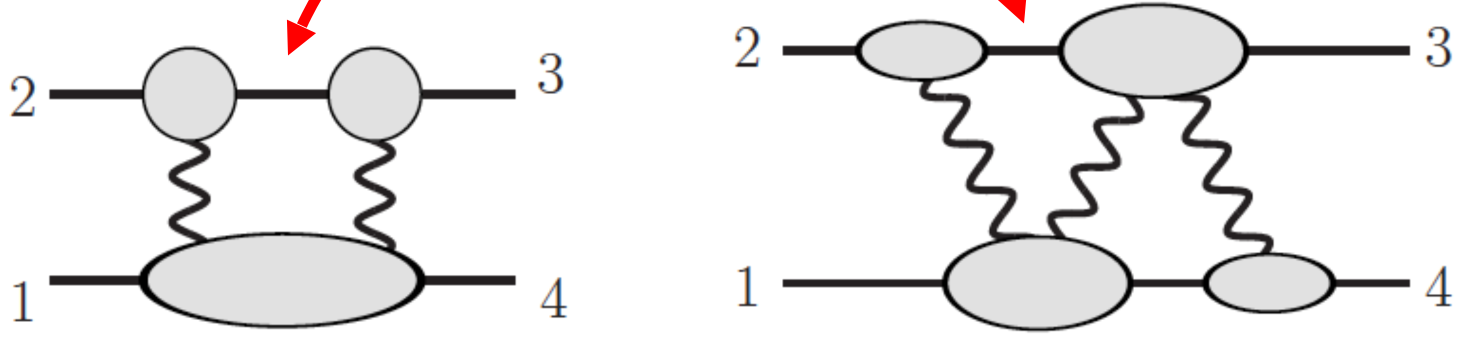


Three-particle cut:

- **Systematic assembly of complete loop amplitudes from tree amplitudes.**
- **Works for any number of particles or loops.**

on-shell

Generalized unitarity as a practical tool for loops.



ZB, Dixon and Kosower;
ZB, Morgan;
Britto, Cachazo, Feng;
Ossala,Pittau,Papadopoulos;
Ellis, Kunszt, Melnikov;
Forde; Badger;
ZB, Carrasco, Johansson, Kosower
and many others

Gravity as a Double copy of Gauge Theory

Kawai, Lewellen, Tye; ZB, Carrasco, Johansson

color factor

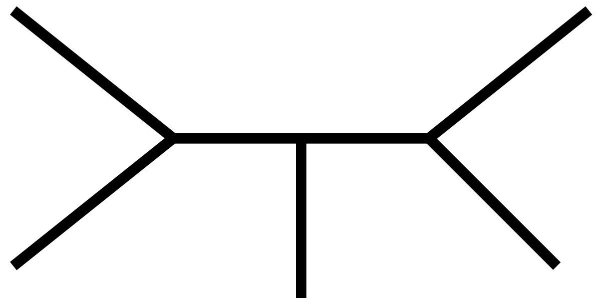
kinematic numerator factor

gauge theory (QCD): $$\mathcal{A}_n^{\text{tree}} = i g^{n-2} \sum_i \frac{c_i\, n_i}{D_i}$$

Feynman propagators

$$c_k = c_i - c_j$$
$$n_k = n_i - n_j$$

$$\boxed{c_i \to n_i}$$



sum over diagrams with only 3 vertices

Einstein gravity: $$\mathcal{M}_n^{\text{tree}} = i \kappa^{n-2} \sum_i \frac{n_i^2}{D_i}$$

$$n_i \sim k_4 \cdot k_5\, k_2 \cdot \varepsilon_1\, \varepsilon_2 \cdot \varepsilon_3\, \varepsilon_4 \cdot \varepsilon_5 + \cdots$$

Gravity and gauge theory kinematic numerators are the same!

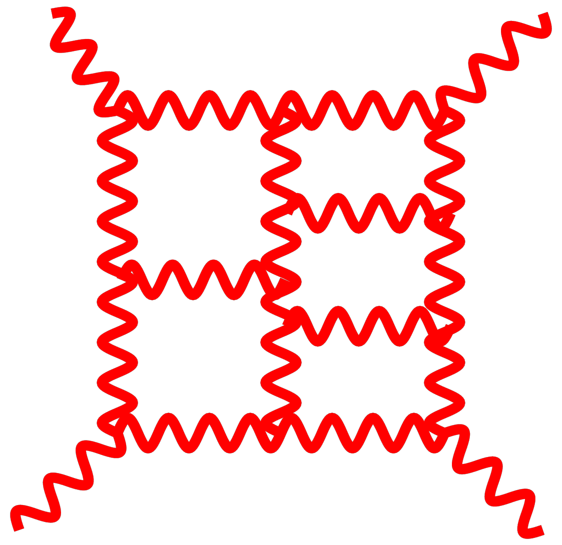
Same ideas conjectured to hold at loop level.

Applications

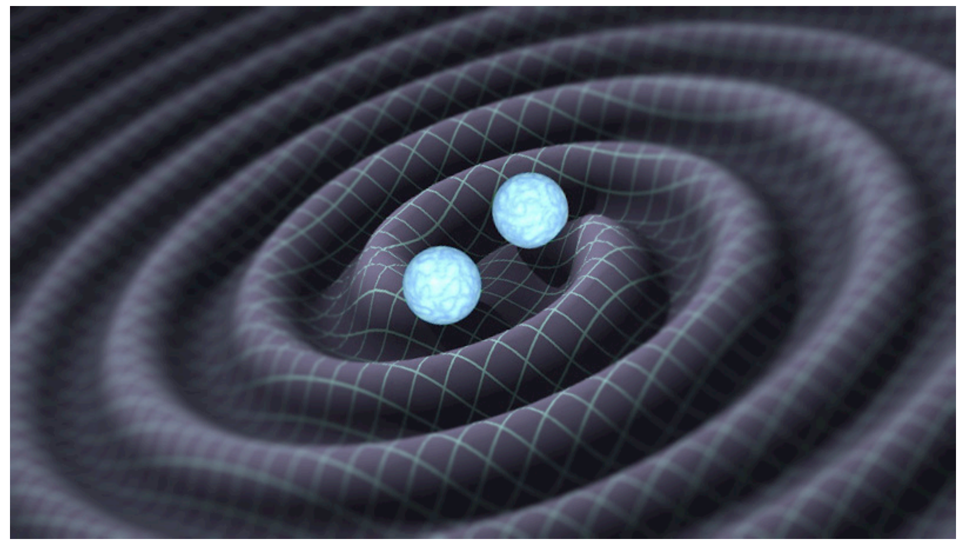
Examples of Applications:

- **5 loop supergravity to study nonrenormalizability of gravity theories.**

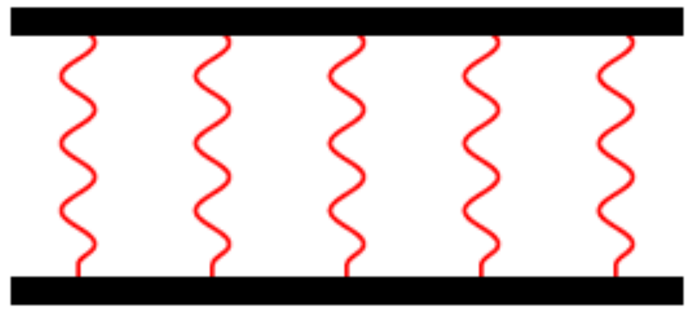
  ZB, Carrasco, Chen, Edison, Johansson, Roiban, Parra-Martinez, Zeng (2018)



- **$G^3 - G^5$ corrections to Newton's potential from GR.**

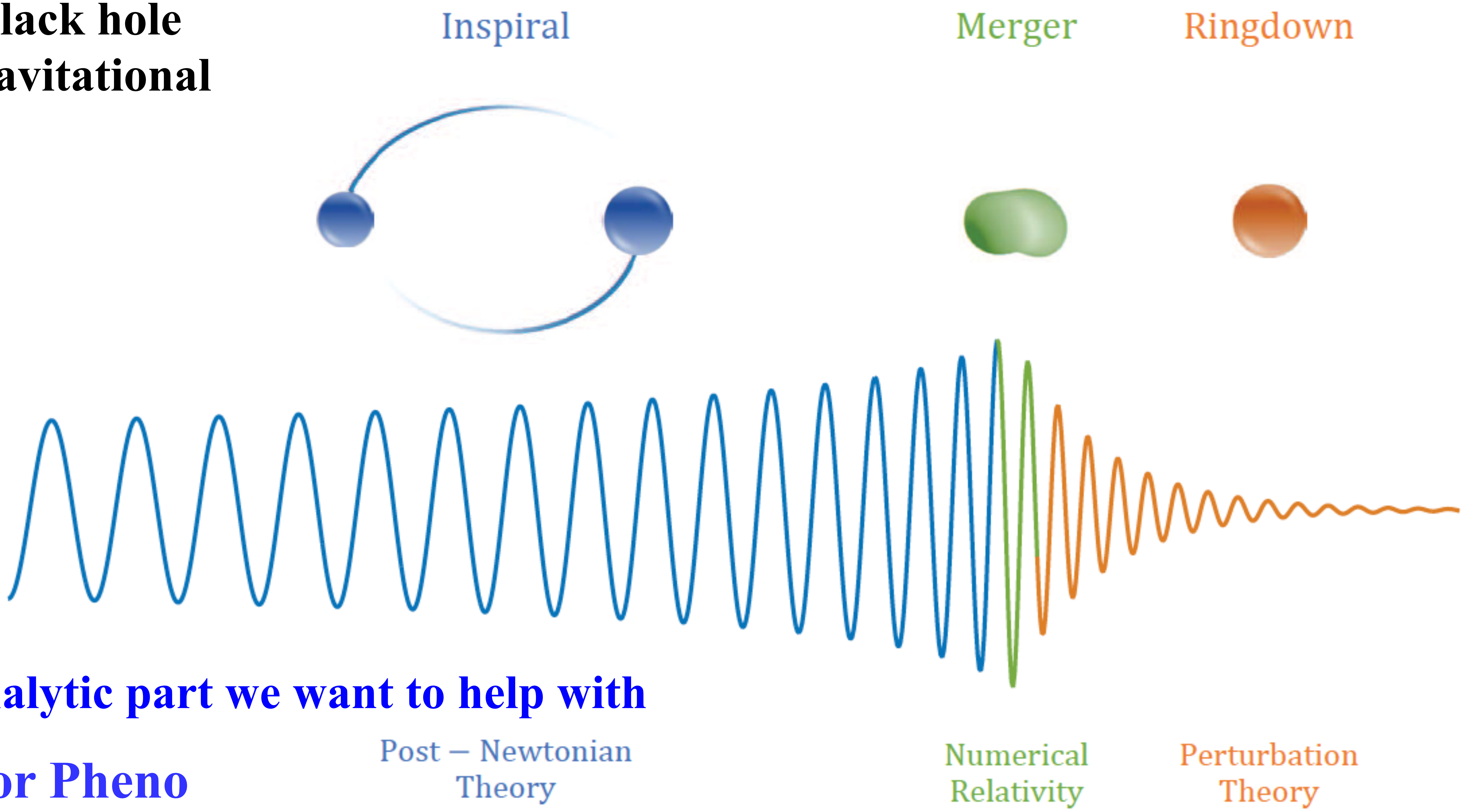


ZB, Cheung, Roiban, Shen, Solon, Zeng (2019)
ZB, Parra-Martinez, Roiban, Ruf. Shen, Solon, Zeng (2021)
ZB, Herrmann, Roiban, Ruf, Smirnov, Smirnov (2024)



Goal: Higher Precision.

Dynamics of black hole inspiral for gravitational waves.



analytic part we want to help with

PN + EOB or Pheno

Small errors accumulate. Need for high precision.

From Antelis and Moreno, arXiv:1610.03567

Basic Approaches

1. Post-Newtonian (**PN**): Expand in G and v
2. Post-Minkowskian (**PM**): Expand in G.
3. Self force (**GSW**): Expand in mass-ratio exact in G. (Semi numerical)
4. Numerical relativity (**NR**): Solve Einstein's equations numerically

- **PM approach fits naturally with scattering amplitudes.**
- **Waveform models import information from all approaches.**

Post Newtonian Approximation

For orbital mechanics:

$$v^2 \sim \frac{GM}{R} \ll 1$$

Expand in G and v^2

virial theorem

In center of mass frame:

$$m = m_A + m_B, \quad \nu = \mu/M,$$

$$\mu = m_A m_B/m, \quad P_R = P \cdot \hat{R}$$

$$\frac{H}{\mu} = \frac{P^2}{2} - \frac{Gm}{R}$$

Newton

$$+ \frac{1}{c^2}\left\{ -\frac{P^4}{8} + \frac{3\,\nu\,P^4}{8} + \frac{Gm}{R}\left(-\frac{{P_R}^2\,\nu}{2} - \frac{3\,P^2}{2} - \frac{\nu\,P^2}{2} \right) + \frac{G^2 m^2}{2R^2} \right\}$$

$$+ \ldots$$

1PN: Einstein, Infeld, Hoffmann; Droste, Lorentz

Hamiltonian known to 4PN order.

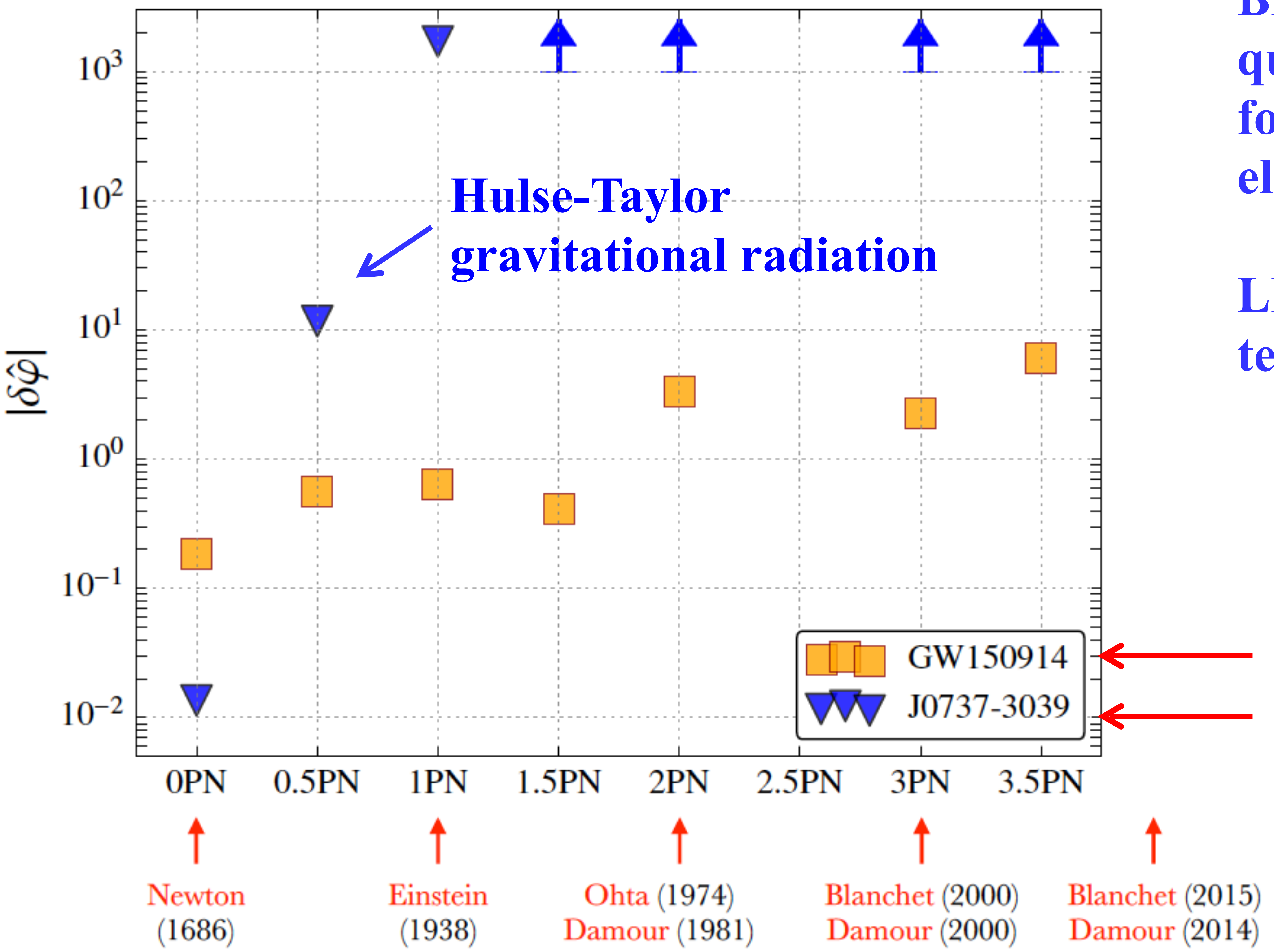
2PN: Ohta, Okamura, Kimura and Hiida (1973); Damour (1982)

3PN: Damour, Jaranowski and Schaefer (2000); L. Blanchet and G. Faye (2000) .

4PN: Damour, Jaranowski and Schaefer (2017); Foffa, Porto, Rothstein, Sturani (2019).

Importance of higher orders for LIGO/Virgo

LIGO/Virgo Collaboration arXiv:1602.03841



Binary pulsar confirms quadrupole radiation formula and not much else.

LIGO/Virgo tests PN terms from GR

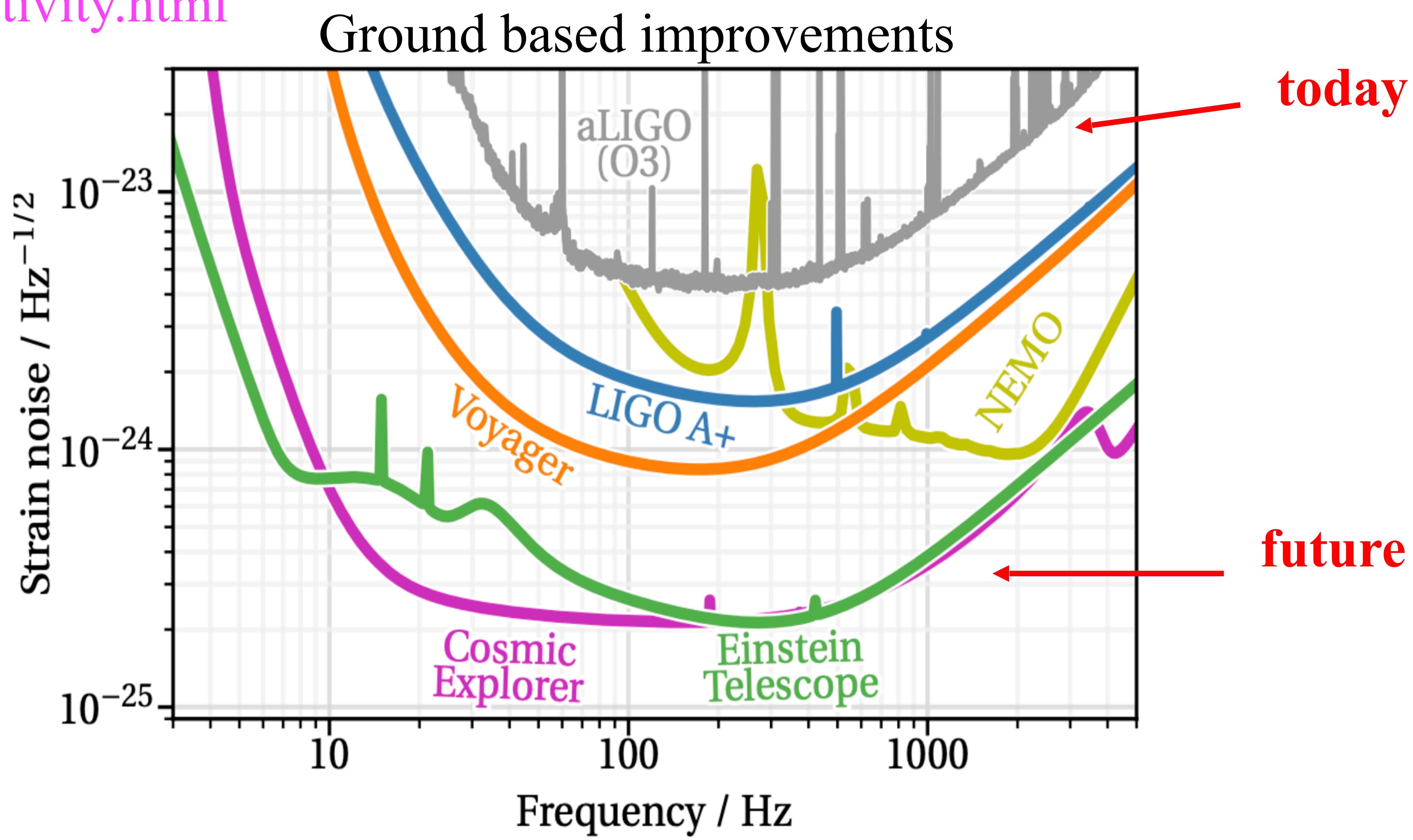
see talk From Harms

LIGO/Virgo sensitive to high PN orders.

Future Detectors

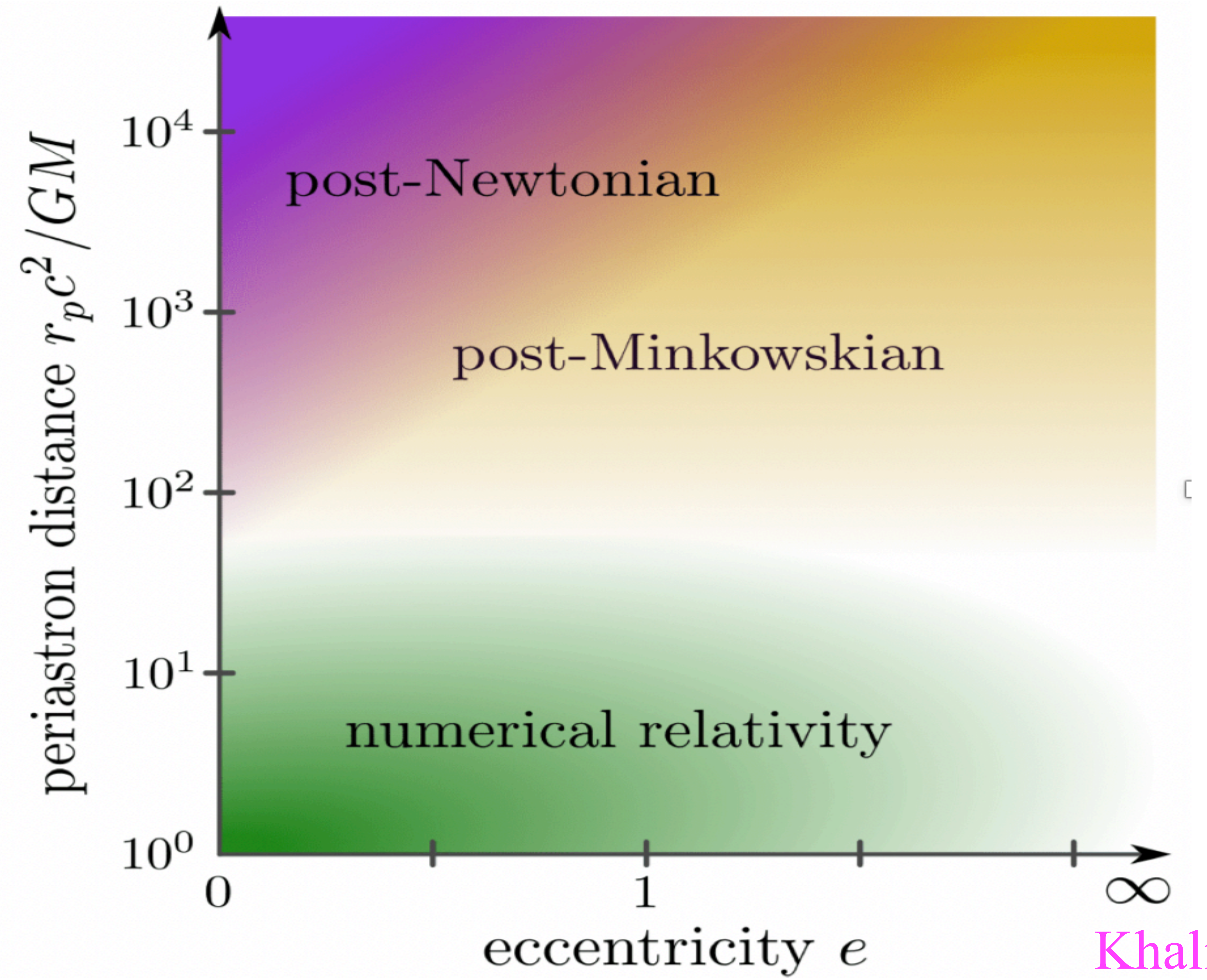
https://cosmicexplorer.org/sensitivity.html

see talk From Harms



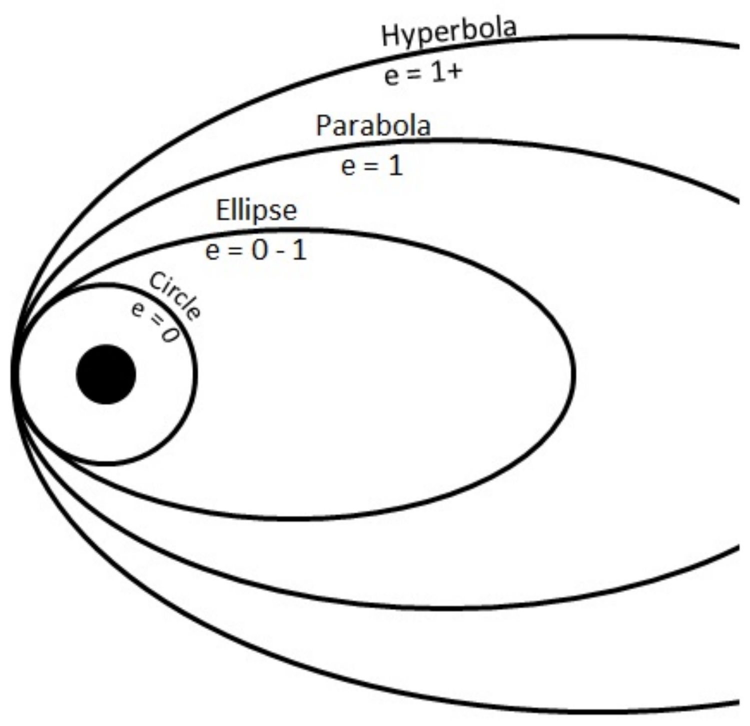
- Depending on parameters, sensitivity improvements up to factor of 100.
- Highly nontrivial theoretical challenge to match upcoming experimental precision.
- Likely need 2 further perturbative orders

Post-Minkowskian Approach



Comments:

- **Unbound orbits cleaner theoretical environment.**
- **Asymptotic flat space for scattering processes**



Khalil, Buonanno, Steinhoff, Vines

Different approaches needed for high precision in all regions. EOB.

Buonanno and Damour

High-energy gravitational scattering and the general relativistic two-body problem

Thibault Damour*
Institut des Hautes Etudes Scientifiques, 35 route de Chartres, 91440 Bures-sur-Yvette, France

(Dated: October 31, 2017)

A technique for translating the classical scattering function of two gravitationally interacting bod-

"… and we urge amplitude experts to use their novel techniques to compute the 2-loop scattering amplitude of scalar masses, from which one could deduce the third post-Minkowskian effective one-body Hamiltonian."

tum gra[…]mplitude of two particles, and we urge amplitude experts to use their novel techniques to compute the 2-loop scattering amplitude of scalar masses, from which one could deduce the third post-Minkowskian effective one-body Hamiltonian

Hard to resist an invitation with this kind of clarity!

The recent observation [1–4] of gravitational wave signals from inspiralling and coalescing binary black holes has been significantly helped, from the theoretical side, by the availability of a large bank of waveform templates, defined [5, 6] within the analytical effective one-body

ntly introduced to derive from the (gauge-invariant) *scattering function* Φ linking (half) the center of mass (c.m.) classical gravitational scattering angle χ to the total energy, $E_{\rm real} \equiv \sqrt{s}$, and the total angular momentum, J, of the system[1]

Very important to us that the GW theory community really needs the results.

numerical simulations of coalescing black-hole binaries (see [12] for a review of perturbative results on binary systems, and [13] for a review of the numerical relativity of binary black holes). Until recently, the perturbative results used to define the EOB conservative dynamics were mostly based on the post-Newtonian (PN) approach to the general relativistic two-body interaction. The conservative two-body dynamics was derived, successively, at the second post-Newtonian (2PN) [14, 15], third post-

$$\frac{1}{2}\chi = \Phi(h, j; \nu)\,, \qquad (1.2)$$

where we denoted

$$h \equiv \frac{E_{\rm real}}{M}\,;\; j \equiv \frac{J}{G m_1 m_2} = \frac{J}{G\mu M}\,, \qquad (1.3)$$

with

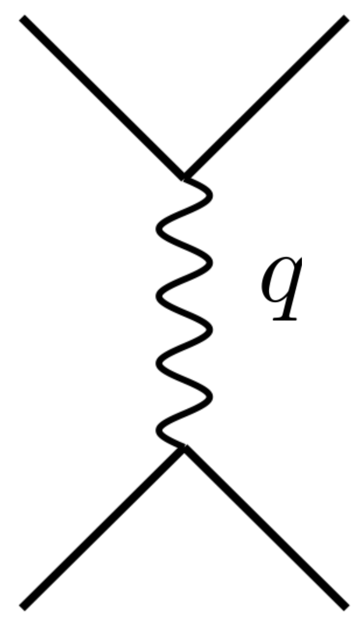
$$M \equiv m_1 + m_2;\; \mu \equiv \frac{m_1 m_2}{\ldots};\; \nu \equiv \frac{\mu}{M} = \frac{m_1 m_2}{(m_1+m_2)^2};$$

2 Body Potentials and Amplitudes

Tree-level: Fourier transform gives classical potential.

$$V(r) \sim \int \frac{d^3q}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{r}} A^{\text{tree}}(\mathbf{q})$$

$$H(\mathbf{p},\mathbf{r}) = \sqrt{\mathbf{p}^2 + m_1^2} + \sqrt{\mathbf{p}^2 + m_2^2} + V(\mathbf{p},\mathbf{r})$$



Corrections to Newtonian potential follows from scattering amplitudes

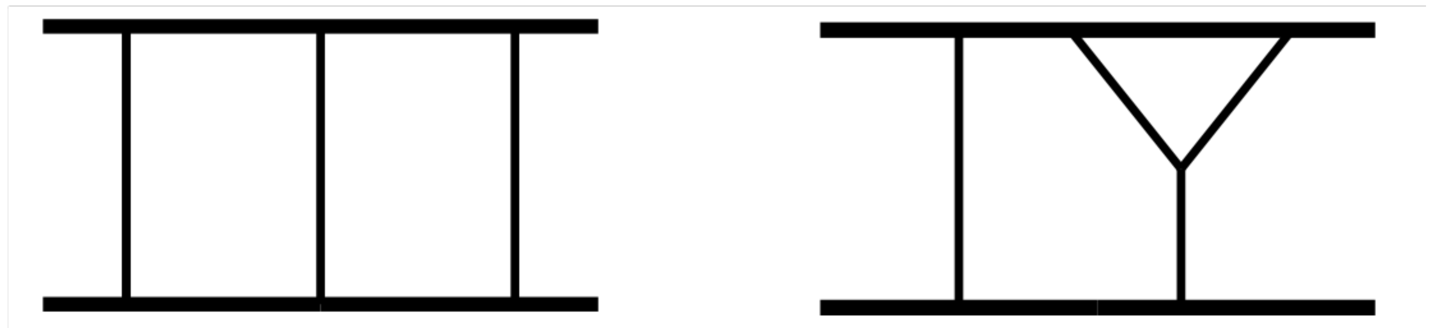
$$V(\mathbf{p},\mathbf{r}) = \sum_{i=1}^{\infty} c_i(\mathbf{p}^2)\left(\frac{G_N}{|\mathbf{r}|}\right)^i$$

Armed with a 2 body Hamiltonian we can do classical physics.

Beyond 1 loop less obvious:

- **Loops have classical pieces.**
- $1/\hbar^L$ **scaling of at** L **loop.**
- **Double counting and iteration.**

$$e^{iS_{\text{classical}}/\hbar}$$



Piece of loops are classical: Our task is to efficiently extract these pieces.

What are we after?

- Replace scattering in General Relativity with a two body Hamiltonian that is easy to use in bound-state problem.
- Extract physics juice, leaving behind complexity of general relativity.

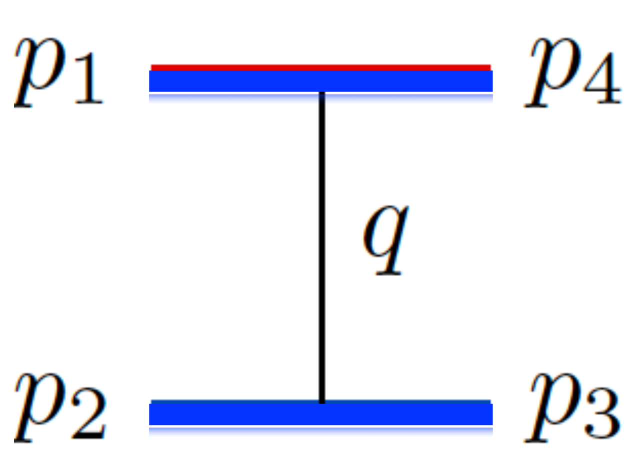
$$V(\mathbf{r}, \mathbf{p}) = -\frac{G m_1 m_2}{r} + \cdots$$

Just like Newton's potential, except

- **Compatible with special relativity (all orders in velocity)**
- **Valid through O(G^5).**
- **In addition, want complete control over radiative contributions.**

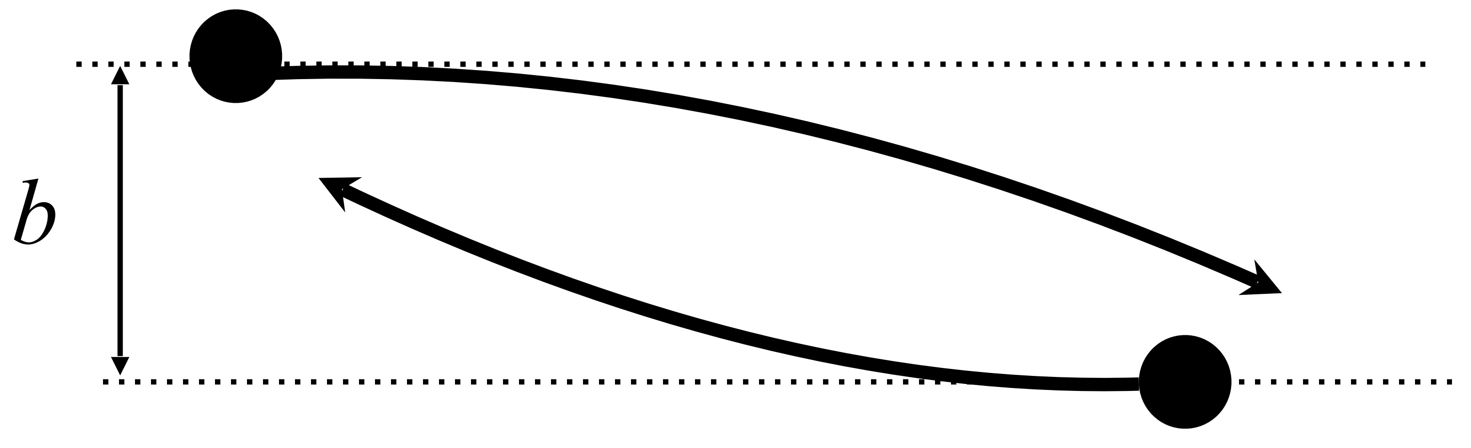
Classical Limit

Consider 2 to 2 scattering



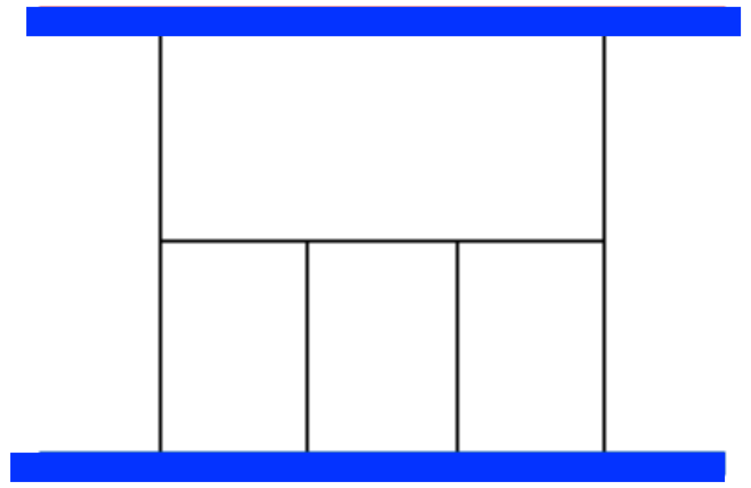
$$s = (p_1 + p_2)^2$$
$$t = (p_1 + p_4)^2$$

$$|q| \sim \frac{1}{|b|}$$



$$s, m_1^2, m_2^2 \sim J^2|t| \gg |t| = |q^2|$$

Large angular momentum limit



Classical contributions live in the soft graviton region

Useful to further subdivide into potential and radiation regions

Beneke and Smirnov

$$\text{potential:} \quad \ell \sim (v, \mathbf{1})|q|\,, \qquad \text{radiation:} \quad \ell \sim (v, \boldsymbol{v})|q|$$

Greatly simplifies the integrals. Eikonal matter propagators

v characteristic velocity

Can also planarize the integrals.

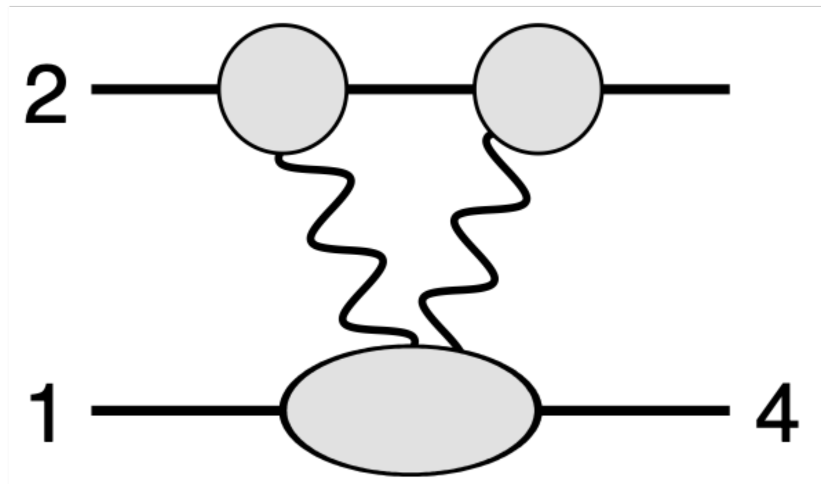
Driesse, Jakobsen, Mogull, Plefka, Sauer, Usovitsch; ZB, Herrmann, Roiban, Ruf, Smirnov, Smirnov, Zeng;

Amplitudes Approach: Unitarity + Double Copy

- **Long-range force: Two matter lines must be separated by on-shell propagators.**
- **Classical potential: 1 matter line per loop is cut (on-shell) to force $E^2 = p^2 + m^2$**

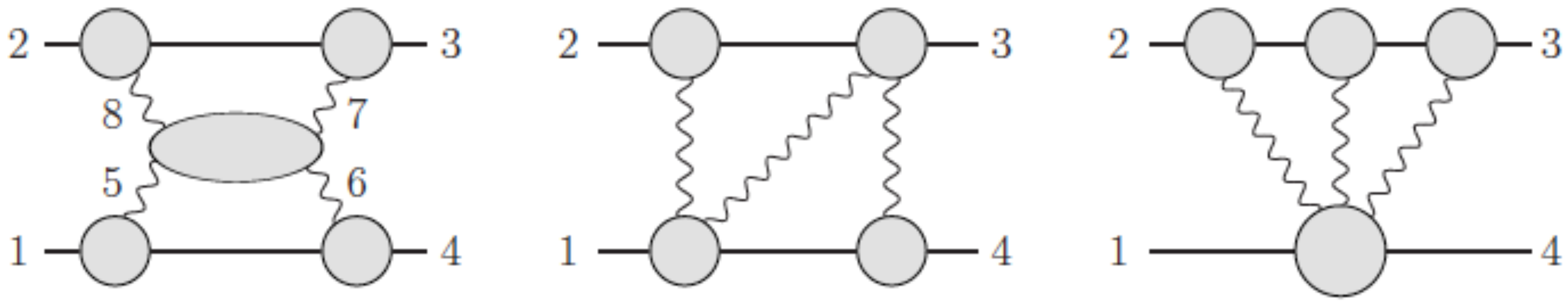
Neill and Rothstein ; Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove; Cheung, Rothstein, Solon

Only independent unitarity cut for O(G^2).

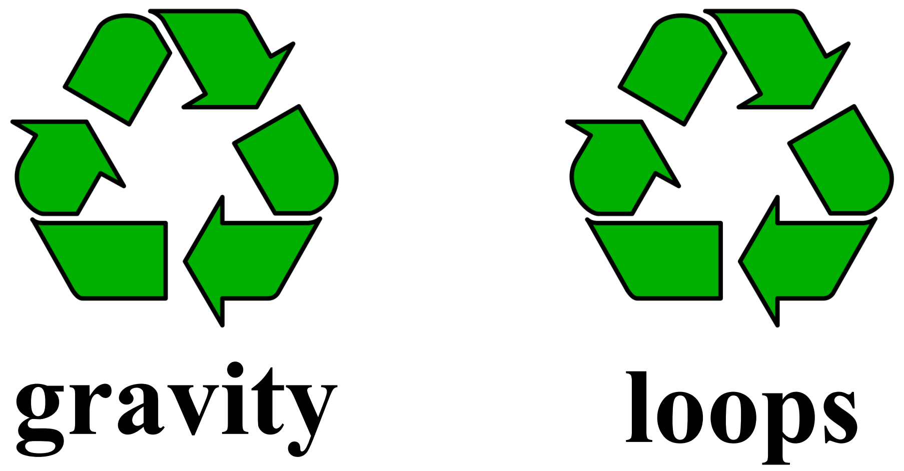


Treat exposed lines on-shell (long range).
Pieces we want are simple!

Independent generalized unitarity cuts for O(G^3).



Amplitude tools fit perfectly with extracting classical pieces we want.



Amplitude in Conservative Classical Potential Limit

ZB, Cheung, Roiban, Shen, Solon, Zeng (BCRSSZ)

The O(G^3) or 3PM conservative terms are:

$$\boxed{\begin{aligned}\mathcal{M}_3 = &\frac{\pi G^3 \nu^2 m^4 \log \boldsymbol{q}^2}{6\gamma^2\xi}\Bigg[3 - 6\nu + 206\nu\sigma - 54\sigma^2 + 108\nu\sigma^2 + 4\nu\sigma^3 - \frac{48\nu\left(3+12\sigma^2-4\sigma^4\right)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}} \\ &- \frac{18\nu\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)}{(1+\gamma)(1+\sigma)}\Bigg] + \frac{8\pi^3 G^3 \nu^4 m^6}{\gamma^4\xi}\Big[3\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)F_1 - 32m^2\nu^2\left(1-2\sigma^2\right)^3 F_2\Big]\end{aligned}}$$

$$m = m_A + m_B, \qquad \mu = m_A m_B/m, \qquad \nu = \mu/m, \qquad \gamma = E/m,$$

$$\xi = E_1 E_2/E^2, \qquad E = E_1 + E_2, \qquad \sigma = p_1 \cdot p_2/m_1 m_2,$$

- **Amplitude remarkably compact.**
- **Arcsinh is new feature.**
- **Derived conservative scattering angle has simple mass dependence.**

Antonelli, Buonanno, Steinhoff, van de Meent, Vines
Comprehensive understanding: Damour

Conservative 3PM Hamiltonian

ZB, Cheung, Roiban, Shen, Solon, Zeng (2019)

The 3PM Hamiltonian:

$$H(\boldsymbol{p}, \boldsymbol{r}) = \sqrt{\boldsymbol{p}^2 + m_1^2} + \sqrt{\boldsymbol{p}^2 + m_2^2} + V(\boldsymbol{p}, \boldsymbol{r})$$

Newton in here

$$V(\boldsymbol{p}, \boldsymbol{r}) = \sum_{i=1}^{3} c_i(\boldsymbol{p}^2) \left(\frac{G}{|\boldsymbol{r}|}\right)^i ,$$

$$c_1 = \frac{\nu^2 m^2}{\gamma^2 \xi} \left(1 - 2\sigma^2\right), \qquad c_2 = \frac{\nu^2 m^3}{\gamma^2 \xi} \left[\frac{3}{4}\left(1 - 5\sigma^2\right) - \frac{4\nu\sigma\left(1 - 2\sigma^2\right)}{\gamma\xi} - \frac{\nu^2(1-\xi)\left(1 - 2\sigma^2\right)^2}{2\gamma^3\xi^2}\right],$$

$$\begin{aligned} c_3 = \frac{\nu^2 m^4}{\gamma^2 \xi} \Bigg[& \frac{1}{12}\left(3 - 6\nu + 206\nu\sigma - 54\sigma^2 + 108\nu\sigma^2 + 4\nu\sigma^3\right) - \frac{4\nu\left(3 + 12\sigma^2 - 4\sigma^4\right)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2 - 1}} \\ & - \frac{3\nu\gamma\left(1 - 2\sigma^2\right)\left(1 - 5\sigma^2\right)}{2(1+\gamma)(1+\sigma)} - \frac{3\nu\sigma\left(7 - 20\sigma^2\right)}{2\gamma\xi} - \frac{\nu^2\left(3 + 8\gamma - 3\xi - 15\sigma^2 - 80\gamma\sigma^2 + 15\xi\sigma^2\right)\left(1 - 2\sigma^2\right)}{4\gamma^3\xi^2} \\ & + \frac{2\nu^3(3 - 4\xi)\sigma\left(1 - 2\sigma^2\right)^2}{\gamma^4\xi^3} + \frac{\nu^4(1 - 2\xi)\left(1 - 2\sigma^2\right)^3}{2\gamma^6\xi^4} \Bigg], \end{aligned}$$

$$m = m_A + m_B, \qquad \mu = m_A m_B / m, \qquad \nu = \mu / m, \qquad \gamma = E/m,$$

$$\xi = E_1 E_2 / E^2, \qquad E = E_1 + E_2, \qquad \sigma = p_1 \cdot p_2 / m_1 m_2,$$

How do we know it is right?

Original checks:

- **Compared to 4PN Hamiltonians after canonical transformation**
- **In test mass limit, $m_1 \ll m_2$, matches Schwarzschild Hamiltonian**

Damour, Jaranowski, Schäfer; Bernard, Blanchet, Bohé, Faye, Marsat

Subsequent calculations confirm our 3PM result:

1. **Subsequent papers confirm our result in 6PN overlap.**

   Blümlein, Maier, Marquard, Schäfer;
   Bini, Damour, Geralico

2. **New calculations reproducing our 3PM result.**

   Cheung and Solon; Kälin, Liu, Porto;

3. **Adding (non-conservative) real radiation clarifies a puzzle at high energies.**

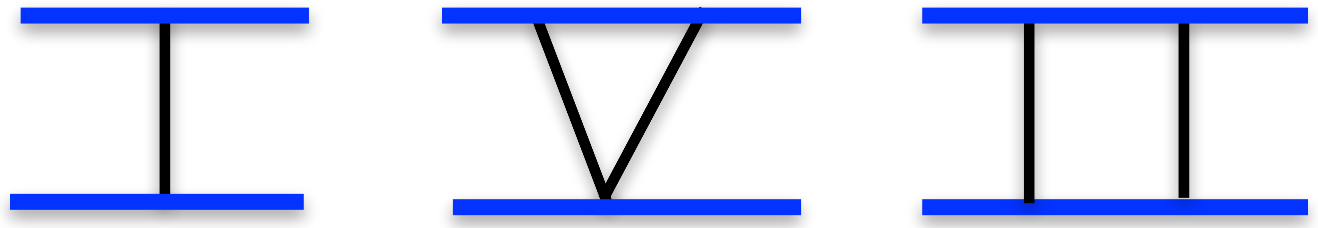
   Di Vecchia, Heissenberg, Russo, Veneziano; Damour

3PM results have passed highly nontrivial checks and careful scrutiny.

Structure of Higher Orders

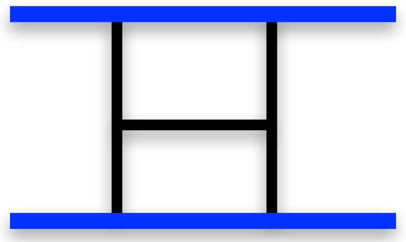
Moving up in orders of PM new effects and features encountered:

1PM and 2PM: **Fixed by geodesic motion, 0SF.**



$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{1 - \frac{2M}{r}} dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta)\, d\phi^2$$

3PM: **Interesting structure in high energy limit. 1SF, m_1/m_2**

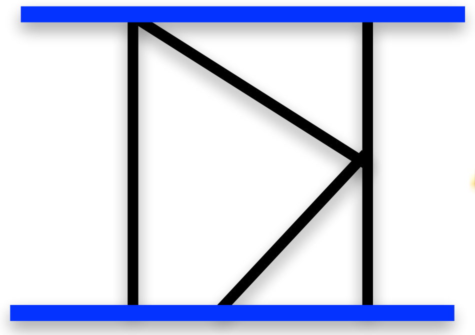


$$\log(E^2/m_1 m_2)$$

Poor high energy behavior cancels against real radiation

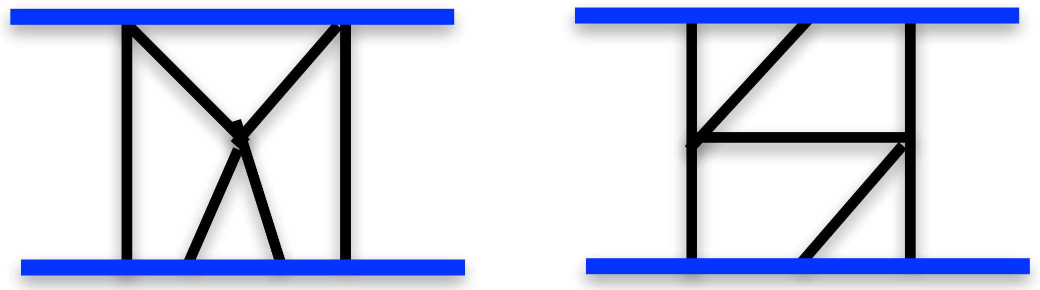
Di Vecchia, Heissenberg, Russo, Veneziano; Damour

4PM: **Tail effect, nontrivial analytic continuations, elliptic integrals, *non*-cancellation of poor high-energy behavior. Nonlocal in time effects.**

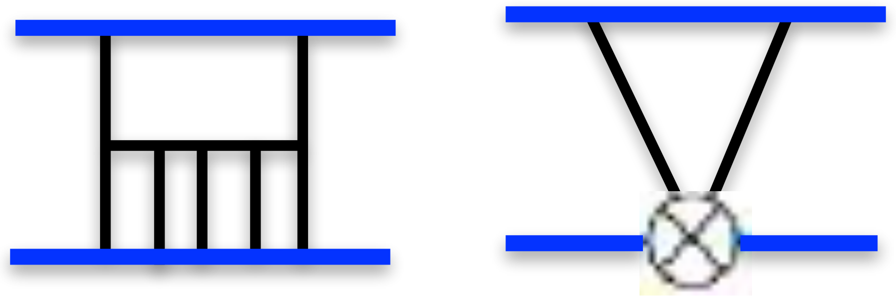


$$\sim K^2\left(\frac{\sigma - 1}{\sigma + 1}\right)$$

5PM: **2SF, Calabi-Yau integrals.**
Nontrivial to separate conservative and dissipative

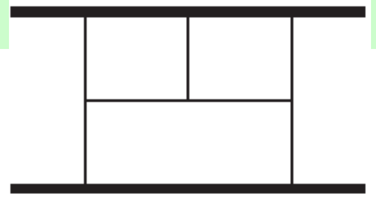


6PM: **Mixing with tidal operators, UV divergences.**
Distinguish BHs from neutron stars.



Conservative Contribution 4PM O(G^4)

ZB, Parra-Martinez, Roiban, Ruf, Shen, Solon, Zeng



O(G^4) amplitude — test particle — 1st self force — Iteration. No need to compute

$$\mathcal{M}_4^{\rm cons} = G^4 M^7 \nu^2 |\boldsymbol{q}| \pi^2 \left[\mathcal{M}_4^{\rm p} + \nu\left(4\mathcal{M}_4^{\rm t}\log\left(\tfrac{p_\infty}{2}\right) + \mathcal{M}_4^{\pi^2} + \mathcal{M}_4^{\rm rem}\right)\right] + \int_\ell \frac{\tilde I_{r,1}^4}{Z_1Z_2Z_3} + \int_\ell \frac{\tilde I_{r,1}^2 \tilde I_{r,2}}{Z_1Z_2} + \int_\ell \frac{\tilde I_{r,1}\tilde I_{r,3}}{Z_1} + \int_\ell \frac{\tilde I_{r,2}^2}{Z_1}$$

$\longleftarrow$ Lower loop, already known, radial action

tail effect

$$D = 4 - 2\epsilon$$

$$\mathcal{M}_4^{\rm t} = r_1 + r_2 \log\left(\tfrac{\sigma+1}{2}\right) + r_3 \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}, \qquad \mathcal{M}_4^{\rm p} = -\frac{35\left(1 - 18\sigma^2 + 33\sigma^4\right)}{8\left(\sigma^2-1\right)}$$

$$\mathcal{M}_4^{\pi^2} = r_4\pi^2 + r_5\,{\rm K}\left(\tfrac{\sigma-1}{\sigma+1}\right){\rm E}\left(\tfrac{\sigma-1}{\sigma+1}\right) + r_6\,{\rm K}^2\left(\tfrac{\sigma-1}{\sigma+1}\right) + r_7\,{\rm E}^2\left(\tfrac{\sigma-1}{\sigma+1}\right),$$ $\longleftarrow$ elliptic

$$\mathcal{M}_4^{\rm rem} = r_8 + r_9\log\left(\tfrac{\sigma+1}{2}\right) + r_{10}\frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} + r_{11}\log(\sigma) + r_{12}\log^2\left(\tfrac{\sigma+1}{2}\right) + r_{13}\frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\log\left(\tfrac{\sigma+1}{2}\right) + r_{14}\frac{\operatorname{arccosh}^2(\sigma)}{\sigma^2-1}$$
$$+ r_{15}\,{\rm Li}_2\left(\tfrac{1-\sigma}{2}\right) + r_{16}\,{\rm Li}_2\left(\tfrac{1-\sigma}{1+\sigma}\right) + r_{17}\frac{1}{\sqrt{\sigma^2-1}}\left[{\rm Li}_2\left(-\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) - {\rm Li}_2\left(\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right)\right].$$

$$\nu = m_1 m_2/(m_1+m_2)^2 \qquad \sigma = p_1\cdot p_2/m_1 m_2,$$

r_{ij} rational coefficients

This is complete conservative contribution.

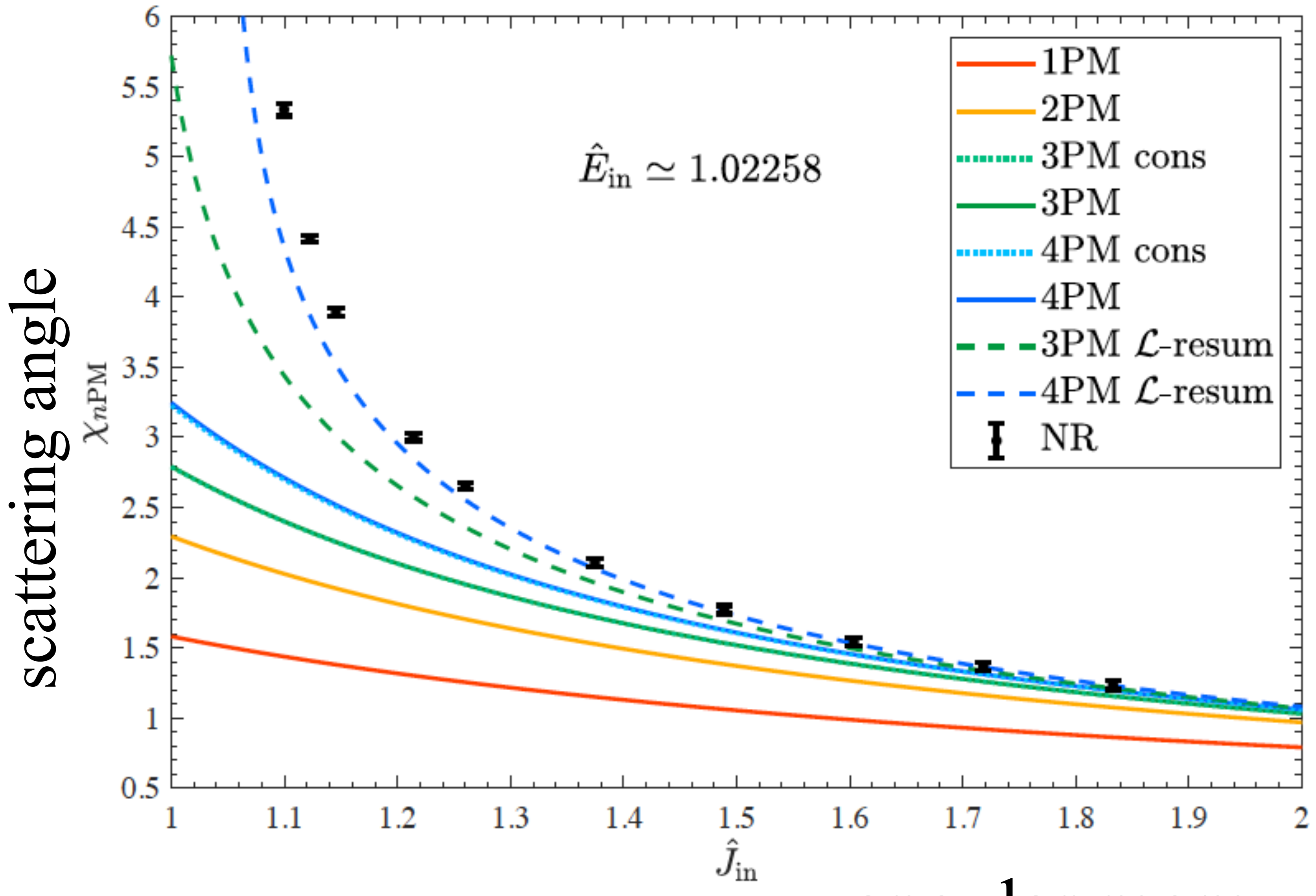
$$\mathcal{M}_4^{\rm radgrav,f} = \frac{12044}{75}p_\infty^2 + \frac{212077}{3675}p_\infty^4 + \frac{115917979}{793800}p_\infty^6 - \frac{9823091209}{76839840}p_\infty^8 + \frac{115240251793703}{1038874636800}p_\infty^{10} + \cdots$$

First 3 terms match 6PN results of Bini, Damour, Geralico.

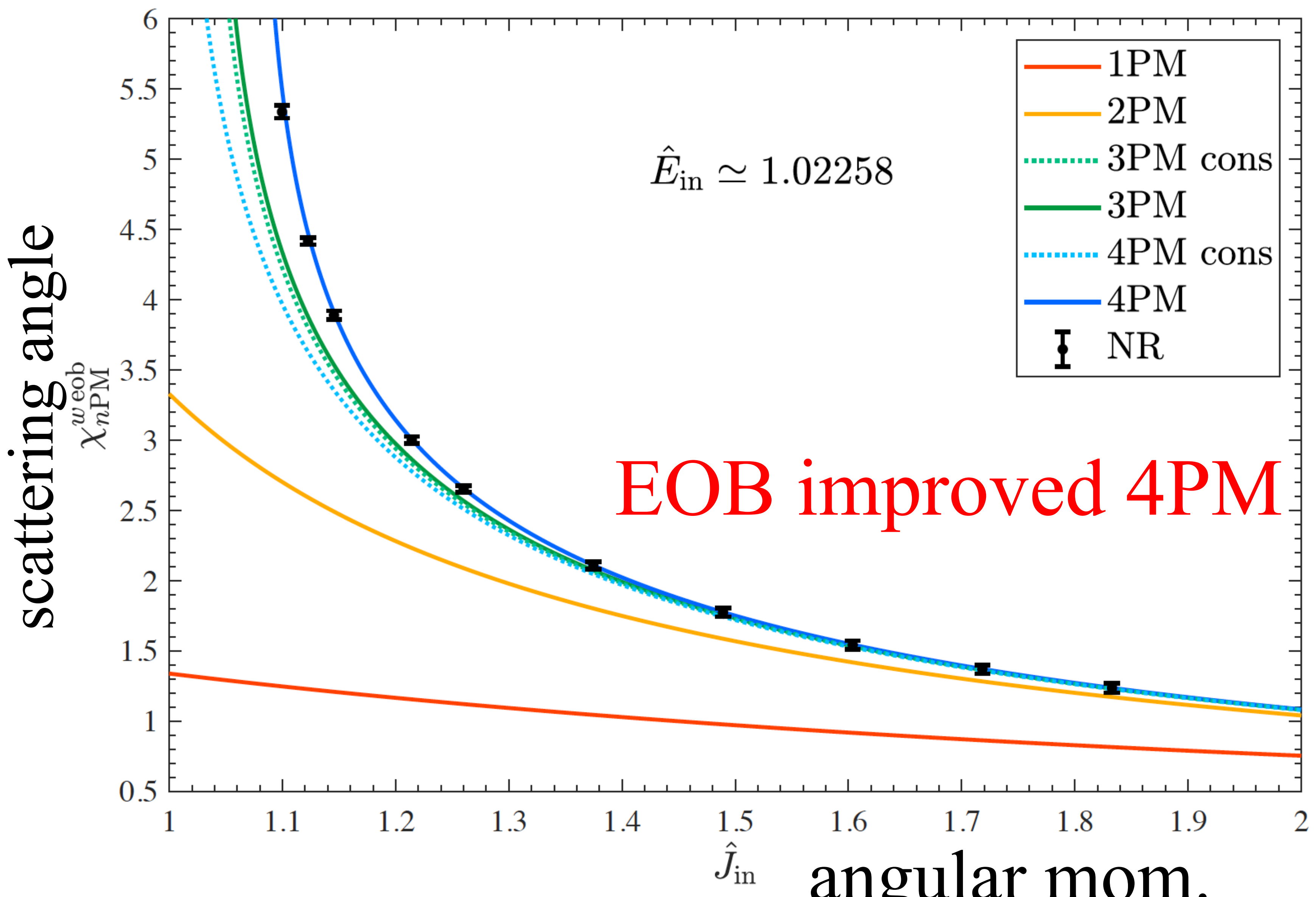
- **Result for angle, including radiation effects completed.** Manohar, Ridgeway, Shen; Dlapa, Kälin, Liu, Porto
- **Potential subtlety remains with PN comparision.** Bluemlein, Maier, Marquard, Schafer; Foffa, Sturani. Luz Almeida, Muller, Foffa, Sturani
- **Analytic continuation to bound case not trivial: tail effect. Recent progress on local part.** Dlapa, Kälin, Liu, Porto; Bini and Damour

Comparison with Numerical Relativity

Khalil, Buonanno, Vines, Steinhoff;   Damour and Rettegno



angular mom.



angular mom.

Plot uses:

[Also comparisons for bound systems, Buonanno], [Mogull], [Patil], [Pompili]

4PM Conservative: ZB, Parra-Martinez, Roiban, Ruf. Shen, Solon, Zeng;
Damgaard, Hansen, Planté, Vanhove; Jakobsen, Gustav Mogull, Plefka, Sauer, Xu;
Bjerrum-Bohr, Plante,Vanhove.

4PM Dissipative: Manohar, Shen and Ridgeway; Dlapa, Kalen, Lui, Neef, Porto;
Damgaard, Hansen, Planté, Vanhove.

NR: Damour,. Guercilena, Hinder, Hopper, Nagar, Rezzolla;

- **Surprisingly good agreement with numerical relativity!**
- **Proves we are on a good track!**
- **Motivates us to go on 5 PM order.**

Conservative Contribution 4PM O(G^4)

ZB, Parra-Martinez, Roiban, Ruf, Shen, Solon, Zeng

O(G^4) amplitude

test particle

tail effect

$$\mathcal{M}_4^{\rm cons} = G^4 M^7 \nu^2 |\boldsymbol{q}| \pi^2 \Big[\mathcal{M}_4^{\rm p} + \nu \overbrace{\Big(4\mathcal{M}_4^{\rm t} \log\left(\tfrac{p_\infty}{2}\right) + \mathcal{M}_4^{\pi^2} + \mathcal{M}_4^{\rm rem}\Big)}^{\text{1st self force}}\Big] + \overbrace{\int_\ell \frac{\tilde{I}_{r,1}^4}{Z_1 Z_2 Z_3} + \int_\ell \frac{\tilde{I}_{r,1}^2 \tilde{I}_{r,2}}{Z_1 Z_2} + \int_\ell \frac{\tilde{I}_{r,1}\tilde{I}_{r,3}}{Z_1}}^{\text{Iteration. No need to compute}} + \int_\ell \frac{\tilde{I}_{r,2}^2}{Z_1}$$

Lower loop, already known,
radial action

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elliptic

$$\mathcal{M}_4^{\rm rem} = r_8 + r_9 \log\left(\tfrac{\sigma+1}{2}\right) + r_{10} \tfrac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} + r_{11} \log(\sigma) + r_{12} \log^2\left(\tfrac{\sigma+1}{2}\right) + r_{13} \tfrac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} \log\left(\tfrac{\sigma+1}{2}\right) + r_{14} \tfrac{\operatorname{arccosh}^2(\sigma)}{\sigma^2-1}$$
$$+ r_{15} \,\mathrm{Li}_2\left(\tfrac{1-\sigma}{2}\right) + r_{16} \,\mathrm{Li}_2\left(\tfrac{1-\sigma}{1+\sigma}\right) + r_{17} \tfrac{1}{\sqrt{\sigma^2-1}} \left[\mathrm{Li}_2\left(-\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) - \mathrm{Li}_2\left(\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right)\right].$$

For dissapative see: Manohar, Ridgeway, Shen;
Dlapa, Kälin, Liu, Porto
Spin-orbit:
Jakobsen, Mogull, Plefka, Sauer, Xu

$$\nu = m_1 m_2/(m_1+m_2)^2 \qquad \sigma = p_1 \cdot p_2 / m_1 m_2,$$

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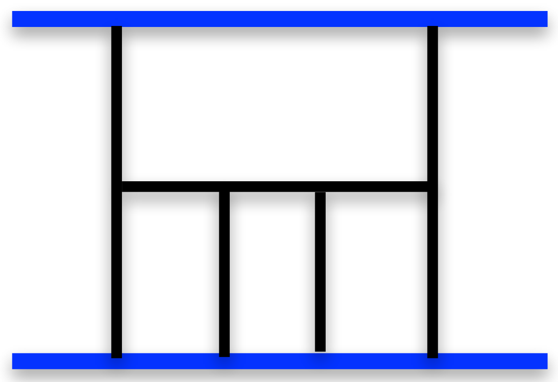
Towards 5PM, $O(G^5)$



Scattering Amplitudes/Worldline

Double copy
Generalized unitarity
Expansion in classical limit

Straightforward

Loop Integrand

Reduction to master integrals
DE's for master integrals
Solutions of DEs.

Hard

Currently working on integration

ZB, Herrmann, Roiban, Ruf, Smirnov, Smirnov, Zeng;
Driesse, Jakobsen, Mogull, Plefka, Sauer, Usovitsch
[Humboldt]

Integrated Amplitude

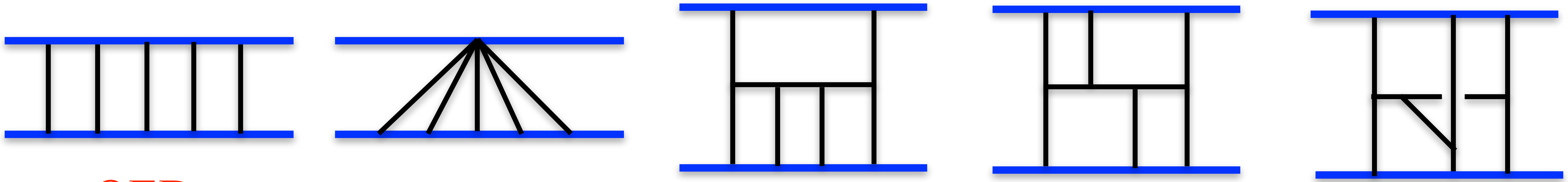
Eikonal, EFT matching computations
Amplitude action relation,
Pick your favorite formalism.

Straightforward

2-Body Hamiltonian or Observables

5PM problem nontrivial, so attack in stages.

Deal With Integration in Stages



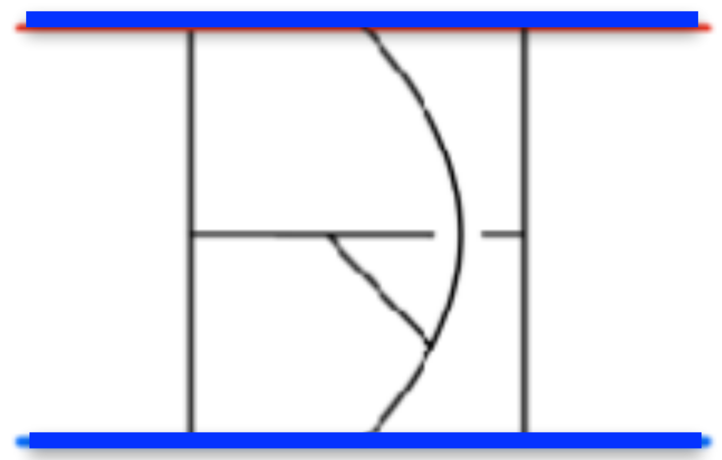
Stages:

1. **QED warmup. Potential mode contributions.** **Done.**
   ZB, Herrmann, Roiban, Ruf, Smirnov, Smirnov, Zeng
2. **1 SF Conservative.** **Done.**
   Driesse, Jakobsen, Mogull, Plefka, Sauer, Usovitsch
3. ***N* = 8 (lower tensor rank)** **1SF potential done, working on 2SF.**
   ZB, Herrmann, Roiban, Ruf, Smirnov, Smirnov, Zeng
4. **2 SF Conservative.** **Harder, but in reach**
5. **Radiative effects.** **Similar**

$$M_{5\mathrm{PM}} = M_{5\mathrm{PM}}^{0\mathrm{SF}} + \nu M_{5\mathrm{PM}}^{1\mathrm{SF}} + \nu^2 M_{\mathrm{PM}}^{2\mathrm{SF}} \qquad \nu = \frac{m_1 m_2}{(m_1+m_2)^2}$$

Learn from each stage to push forward the next one.

Bottleneck: Integral Reduction



Primary bottleneck is integration by parts:

22 indices: 13 propagator and 9 irreducible scalar products (ISPs)

$$\int \frac{d^{4D}k\,[u_2\cdot k_1]^{a_{-14}}[u_2\cdot k_4]^{a_{-15}}[u_1\cdot k_2]^{a_{-16}}[u_1\cdot k_3]^{a_{-17}}[k_1\cdot q]^{a_{-18}}[k_2\cdot q]^{a_{-19}}[k_1\cdot k_2]^{a_{-20}}[k_1\cdot k_4]^{a_{-21}}[k_2\cdot k_3]^{a_{-22}}}{[-2u_2\cdot k_2]^{a_1}[-2u_2\cdot k_{123}]^{a_2}[2u_1\cdot k_{234}]^{a_3}[2u_1\cdot k_{1234}]^{a_4}[k_1^2]^{a_5}[k_2^2]^{a_6}[k_3^2]^{a_7}[k_{13}^2]^{a_8}[k_4^2]^{a_9}[k_{34}^2]^{a_{10}}[k_{234}^2]^{a_{11}}[(k_{123}-q)^2]^{a_{12}}[(k_{1234}-q)^2]^{a_{13}}}$$

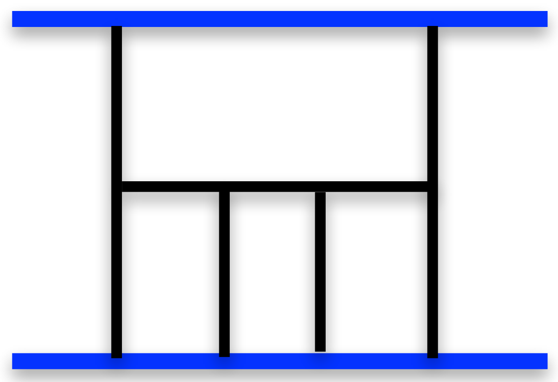
- **Can encounter up to 8 numerator powers and 4 doubled propagators.**
- **FIRE and KIRA need to be carefully tuned.**
- **Private code specialized for finite field numerical techniques.**

$$0 = \int \prod_i^L \frac{d^D\ell_i}{(2\pi)^D}\frac{\partial}{\partial \ell_i^\mu}\frac{N^\mu(\ell_k,p_i)}{D_1^{a_1}\cdots D_n^{a_n}}$$

ZB, Herrmann, Roiban, Ruf, Smirnov, Smirnov, Zeng; Driesse, Jakobsen, Mogull, Plefka, Sauer, Usovitsch

High-Loop Integration

In QCD/Amplitudes advanced technology for loop integrals which we import.



1. **IBP greatly simplified in classical limit.** Chetyrkin, Tkachov; Laporta; Henn; Henn and Smirnov Beneke and Smirnov; Parra-Martinez, Ruf, Zeng
2. **Choose master integrals to simplify the DEs and IBP.** A. Smirnov and V. Smirnov; Usovitsch
3. **Use finite prime fields and reconstruction for toughest IBPs.** Manteuffel, Schabinger; Peraro
4. **Set up a DEs for master integrals.** Kotikov, ZB, Dixon and Kosower; Gehrmann, Remiddi

IBP: $$0 = \int \prod_i^L \frac{d^D \ell_i}{(2\pi)^D} \frac{\partial}{\partial \ell_i^\mu} \frac{N^\mu(\ell_k, p_M)}{Z_1 \ldots Z_n}$$

Solve linear relations for master integrals

DEs: $$\partial_x \vec{I} = A(x, \epsilon) \vec{I} \, ,$$

Solve DEs either as series or basis of functions.

Many tools available: We use upgraded FIRE, a private finite field IBP program, LiteRed, FiniteFlow.

Smirnov, Chuharev; Lee; Peraro

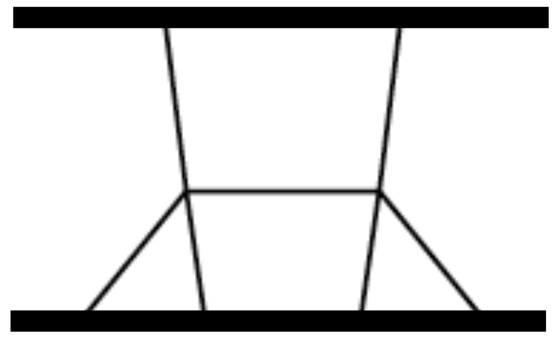
Another important tool is an upgraded version of KIRA

Maierhoefer, Usovitsch, Uwer;
Klappert, Lange, Maierhöfer, Usovitsch

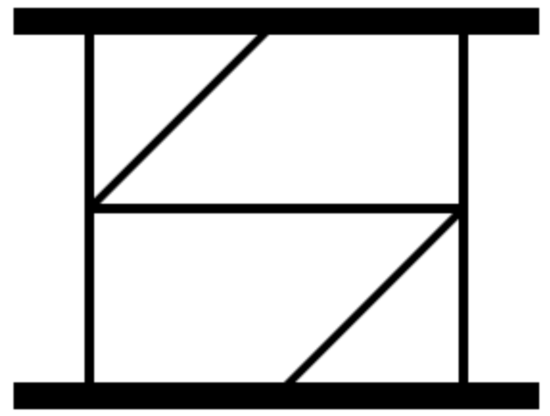
Iterated integrals

Elliptic and Calabi-Yau integrals appear in master integrals.

Frellesvig, Morales, Wilhelm; Klemm, Nega, Sauer, Plefka
ZB, Herrmann, Roiban, Ruf, Smirnov, Smirnov, Zeng.



At 1 SF, Elliptic integrals.



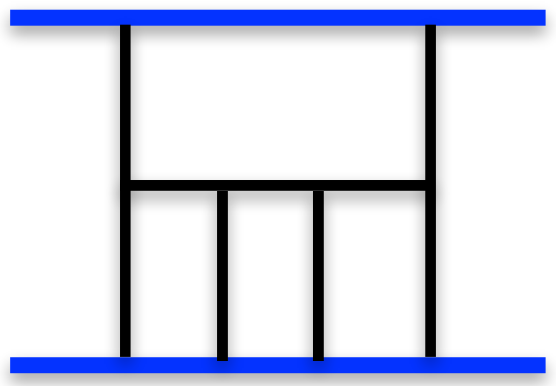
At 2 SF, Calabi-Yau 3-fold integrals.

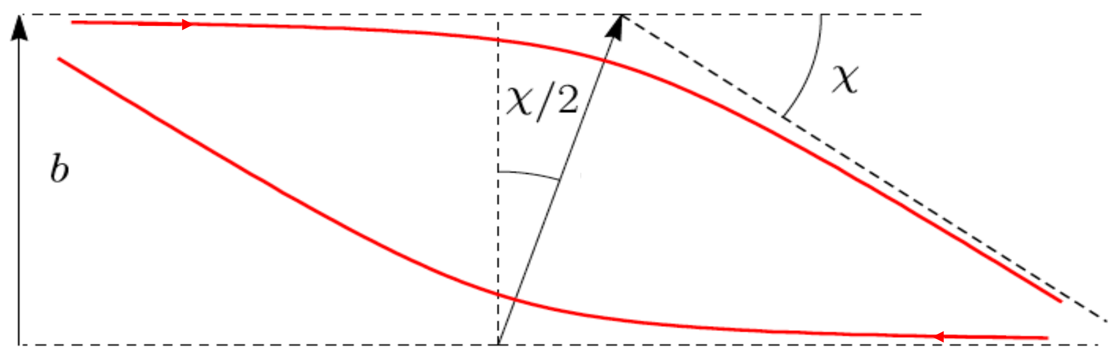
- **At 1SF no elliptic integrals in final result!**
- **Will this simplicity continue to 2 SF sector?**

Opportunity for those interested in the mathematics of Feynman integrals

5PM Scattering Angle $N = 8$ Supergravity (1SF Potential)

ZB, Herrmann, Roiban, Ruf, Smirnov, Smirnov, Zeng





$$\tilde{I}_{r,5}^{\rm 1SF,fin.} = r_1 + r_2\,\mathrm{F}_0 + r_3\,\mathrm{F}_0^2 + r_4\,\mathrm{F}_1 + r_5\,\mathrm{F}_2\,.$$

$$\mathrm{F}_0 = \frac{2x}{1-x^2}\ln(x)\,,$$
$$\mathrm{F}_1 = \frac{2x}{1-x^2}\left[-\mathrm{Li}_2(1-x) - \mathrm{Li}_2(-x) - \ln(x)\ln(x+1) - \frac{1}{2}\zeta_2\right],$$
$$\mathrm{F}_2 = \frac{2x}{1-x^2}\left[-\mathrm{Li}_2(1-x) + \mathrm{Li}_2(-x) - \frac{1}{2}\ln^2(x) + \ln(x)\ln(x+1) + \frac{1}{2}\zeta_2\right]$$

Remarkably simple, elliptic integrals cancel.

$$r_1 = \frac{16c_\phi^3}{\sigma^2-1}\left[-\frac{\sigma\left(5\sigma^2-4\right)c_\phi^3}{5\left(\sigma^2-1\right)^3} - \frac{2c_\phi^2}{\sigma^2-1} + 8\right],$$
$$r_2 = 32c_\phi^2\left[-\frac{\sigma^2c_\phi^4}{\left(\sigma^2-1\right)^3} - \frac{4\sigma c_\phi^3}{\left(\sigma^2-1\right)^2} - \frac{9\left(2\sigma^2-1\right)c_\phi^2}{2\left(\sigma^2-1\right)^2} - \frac{8\sigma c_\phi}{\sigma^2-1} + 1\right],$$
$$r_3 = 16c_\phi\left[-\frac{\sigma^3c_\phi^5}{\left(\sigma^2-1\right)^3} - \frac{6\sigma^2c_\phi^4}{\left(\sigma^2-1\right)^2} + \frac{6\sigma\left(2-3\sigma^2\right)c_\phi^3}{\left(\sigma^2-1\right)^2} + \frac{\left(8-32\sigma^2\right)c_\phi^2}{\sigma^2-1}\right.$$
$$\left.-\frac{92\sigma c_\phi}{3} - \frac{40}{3}\left(\sigma^2-1\right)\right],$$
$$r_4 = 32c_\phi\left[-\frac{\sigma c_\phi^3}{\left(\sigma^2-1\right)^2} - \frac{2c_\phi^2}{\sigma^2-1} - \frac{4\sigma c_\phi}{3\left(\sigma^2-1\right)} - \frac{8}{3}\right],$$
$$r_5 = 64c_\phi\left[\frac{\sigma^2c_\phi^3}{\left(\sigma^2-1\right)^2} + \frac{4\sigma c_\phi^2}{\sigma^2-1} + \frac{2\left(7\sigma^2-6\right)c_\phi}{3\left(\sigma^2-1\right)} + \frac{4\sigma}{3}\right].$$

See also GR 1SF conservative results.

Driesse, Jakobsen, Mogull, Plefka, Sauer, Usovitsch

Scattering angle: $\chi = -\dfrac{\partial I_r}{\partial J}$

It's just a matter of time before 5PM is complete.

Other Directions for Amplitudes Methods

Applying amplitudes methods to gravitational-wave physics is now well developed.

- **Pushing state of the art for high orders in G. Now pushing to G^5**
  ZB, Cheung, Roiban, Parra-Martinez, Ruf, Shen, Solon, Zeng; Di Vecchia, Heissenberg, Russo, Veneziano; Damgaard, Hansen, Plante, Vanhove; Dlapa, Kälin, Liu, Porto Ridgway, Shen; ZB, Herrmann, Roiban, Ruf, Smirnov, Smirnov, Zeng; Driesse, Jakobsen, Mogull, Plefka, Sauer, Usovitsch; Edison and Levi; etc
- **Waveforms** Cristofoli, Gonzo, Kosower, O'Connell; Herrmann, Parra-Martinez, Ruf, Zeng; Di Vecchia, Heissenberg, Russo, Veneziano; Herderschee, Roiban, Teng; Georgoudis, Heissenberg , Vazquez-Holm; Brandhuber, Brown, Chen, De Angelis, Gowdy , Travaglini; etc
- **Finite-size effects** Cheung and Solon; Haddad and Helset; Kälin, Liu, Porto; Cheung, Shah, Solon; ZB, Parra-Martinez, Roiban, Sawyer, Shen, etc
- **Spin** Vaidya; Geuvara, O'Connell, Vines; Chung, Huang, Kim, Lee; ZB, Luna, Roiban, Shen, Zeng; Kosmopoulos, Luna; Febres Cordero, Kraus, Lin, Ruf, Zeng; Aoude, Haddad, Helset; ZB, Kosmopoulos, Luna, Roiban, Teng; Kim, Steinhoff; Aoude and Ochirov; Ben-Shahar; Vine, Sheopner; Gatica; Jakobsen, Mogull, Plefka, Sauer Cangemi, Chiodaroli,Johansson,Ochirov, Pichini,Skvortsov; Luna,Moynihan, O'Connell,Ross; Bautista, Guevara,Kavanagh, Vines; Bautista, Bonelli, Iosa, Tanzini, Zhou; Buonanno, Mogull, Patil, Pompili; etc
- **Absorption** Goldberger and Rothstein; Aoude, Ochirov; Jones, Ruf; Chen, Hsieh, Huang, Kim; etc

Many great young people!

Conclusions

Scattering amplitudes offer new perspective on gravitational-wave theory.

1. **Novel way to look at perturbative gravity.**
   — **On-shell approach. On-shell recursion and unitary method.**
   — **Everything flows from graviton being a massless spin-2 particle.**
   — **Double copy shows gravity follows from gauge theory.**

2. **Sample Application: High orders, marching on 5PM.**

Judging from the pace of progress expect many new results in coming years.
SMEFT, sYM, supergravities, and of course gravitational waves.

Extra

Methods for Extracting Classical Physics.

There are now multiple alternative ways to extract classical physics.

- **EFT matching to 2 body Hamiltonian** — Cheng, Solon, Rothstein; ZB, Cheung, Roiban, Shen, Zeng
- **Map to EOB** — Bini, Damour, Geralico
- **Calculate physical observables** — Kosower, Maybee, O'Connell
- **Eikonal phase** — Amati, Ciafaloni, Veneziano; Di Vecchia, Heissenberg, Russo, Veneziano
- **Amplitude radial-action relation** — ZB, Parra-Martinez, Roiban, Ruf, Shen, Solon, Zeng
- **Exponential representation** — Damgaard, Plante, Vanhove; Bjerrum-Bohr, Plante, Vanhove
- **Heavy mass field theory** — Brandhuber, Chen, Travaglini, Wen; Damgaard, Haddad, Helset
- **World line formalisms** — Goldberger, Rothstein; Levi, Steinhoff; Dlapa, Kälin, Liu, Porto; Jakobson, Mogul, Plefka, Steinhoff; Edison, Levi; etc

For pushing into new territory we still prefer EFT matching.

Effective Field Theory is a Clean Approach

Build EFT from which we can read off potential.
Want a Newtonian-like potential,
with GR corrections

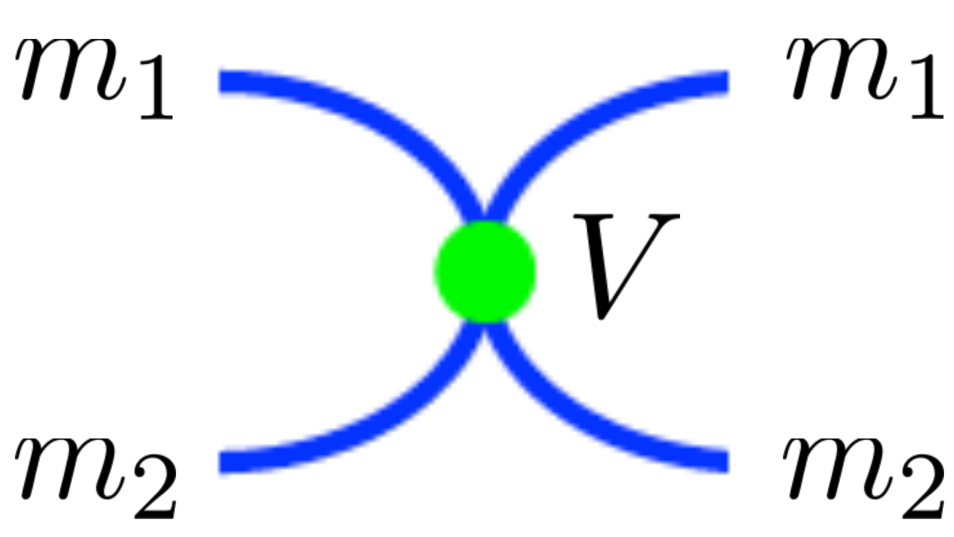
Goldberger and Rothstein
Neill, Rothstein
Cheung, Rothstein, Solon (2018)

$$L_{\text{kin}} = \int_{\boldsymbol{k}} A^\dagger(-\boldsymbol{k}) \left(i\partial_t + \sqrt{\boldsymbol{k}^2 + m_A^2} \right) A(\boldsymbol{k}) + \int_{\boldsymbol{k}} B^\dagger(-\boldsymbol{k}) \left(i\partial_t + \sqrt{\boldsymbol{k}^2 + m_B^2} \right) B(\boldsymbol{k})$$

***A*, *B* scalars represents spinless black holes**

$$L_{\text{int}} = -\int_{\boldsymbol{k},\boldsymbol{k}'} V(\boldsymbol{k},\boldsymbol{k}')\, A^\dagger(\boldsymbol{k}') A(\boldsymbol{k}) B^\dagger(-\boldsymbol{k}') B(-\boldsymbol{k})$$

potential we want to obtain



$$H(\boldsymbol{p},\boldsymbol{r}) = \sqrt{\boldsymbol{p}^2 + m_1^2} + \sqrt{\boldsymbol{p}^2 + m_2^2} + V(\boldsymbol{p},\boldsymbol{r})$$

2 body Hamiltonian in c.o.m. frame.

Match amplitudes of this theory to the full theory in classical limit to extract a classical potential of the type Newton would like.

Our gravitational-wave theory friends want Hamiltonians.